Last course:

Polyhedra $X \overset{\text{homoemorphic}}{\sim}$ Sphere.

$\Rightarrow \quad v - e + f = 2.$

vertices edges faces

Legendre's Pf:

$\Omega = S^n.$

$$S(\Delta) = \sum_{i=1}^{n} \alpha_i - (n-2)\pi$$

Area of $S^2 = 4\pi$

$$\|$$

$$\sum_i \left(\sum_j \alpha_{ij} - n\pi + 2\pi \right).$$

$$\|$$

$$2\pi \cdot V - 2\pi \cdot e + 2\pi \cdot f$$

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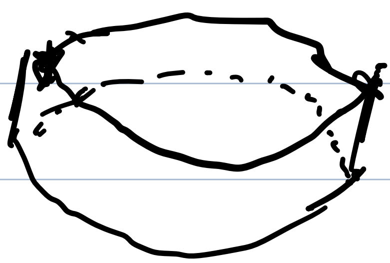
1) 

(2) Mobius: a half twist.

(3). a full twist.

(1), (3) is same for an ant.

Conclusion:

understand homeomorphism using the space itself, but not it's embedding into higher space.

$f: X \to Y$, X, Y subsets of $\mathbb{R}^n$. f is continous

$\Leftrightarrow$ inverse image of open is open.

Neighboorhood.

(a). $\forall x \in X$, $\forall N \in \mathcal{N}(X, x)$.

$x \in N$

(b) closed under finite intersection.

(c). $N \in \mathcal{N}(X, x)$.

$u \supseteq N$

$\Rightarrow u \in \mathcal{N}(X, x)$.

(d). $\mathring{N} = \left\{ z \in N \,\middle|\, N \in \mathcal{N}(X, z) \right\}$.

then $\mathring{N} \in \mathcal{N}(X, x)$.

Ex. $Y \subseteq X$.

$$\begin{tikzcd} X \text{ metric space} \arrow[r] \arrow[d] & X \text{ topology} \arrow[d] \\ d|_Y : \text{metric} \arrow[r, "\sim", "\text{homeomorphic.}"'] & Y \text{ subspace topology} \end{tikzcd}$$

Example. $X = \mathbb{R}$ $\forall x \in X$.

We define $\mathcal{N}(X, x) = \{ V \mid x \in V, X \setminus V$ finite$\}$.

called cofinite topology / Zariski topology (for affine line).

$f : (\mathbb{R}, \text{metric}) \to (\mathbb{R}, \text{Zariski})$

Identity map.

f is one-to-one, onto and continous.

which means Zariski topology

is weaker than metric topology.

Example:

$X = [0,1)$

$Y = S^1$

$f: x \to e^{2\pi x i}$

bijection.

f continuous, f^{-1} is not

3. $Y \subseteq X$ $\quad X$ metric space

prove the subspace topology of Y and the metric map restricts on Y are the same.

subspace topology: $\mathcal{N}_1(Y, y) =$

$$\{ S \mid \exists S' \in \mathcal{N}(X, x),\ S' \cap Y = S \}.$$

metric topology: $\mathcal{N}_2(Y, y) =$

$$\{ S \mid \exists r > 0,\ s.t.\ B_r(y) \subseteq S \}.$$

$\mathcal{N}_1 \subseteq \mathcal{N}_2$:

If $S \in N_1$, suppose $S = S' \cap Y$

$B'_r(y) \subseteq S'$

$\Rightarrow B_r(y) = B'_r(y) \cap Y \subseteq S$

$\Rightarrow S \in N_2$

$N_2 \subseteq N_1$:

If $S \in N_2$, suppose $B_r(y) \subseteq S$

Let $S' = S \sqcup (X \setminus Y)$.

then $B'_r(y) \subseteq S'$ and $S' \cap Y = S$

$\Rightarrow S \in N_1$

4. two strip with a full twist.

- $N \in \mathcal{N}(X, x)$

$\mathring{N} = \{ z \mid z \in N.\ N \in \mathcal{N}(X, z) \}$.

$\Rightarrow (\mathring{N})^{\circ} = \mathring{N}$

It's natural to define an open set associated to N

By say O is open ($O \in \mathcal{O} \subseteq 2^X$).

$\Leftrightarrow \mathring{O} = O$.

We now verify:

(1) $\forall x \in X$, $\mathcal{N}(X,x) \neq \phi$

$\Rightarrow X \in \mathcal{N}(X,x) \Rightarrow X \in \mathcal{O}$

(2) $\phi \in \mathcal{O}$

(3) if $O = \bigcup_{i \in I} O_i$, $x \in O$

$\Rightarrow \exists i$, $x \in O_i$

$\Rightarrow O \in \mathcal{N}(X,x)$

(4) $O_1, O_2 \in \mathcal{O}$

$\Rightarrow O_1 \cap O_2 \in \mathcal{O}$.

by induction $\mathcal{O}_i \subseteq \mathcal{O}_1$

Def. A topology defined by $(X, \mathcal{O})$

means $\mathcal{O} \subseteq 2^X$, s.t.

(1) $\phi, X \in \mathcal{O}$

(2) closed under arbitrary union.

(3) closed under finite intersection

Def $\mathcal{N}(X, x) := \{ Y \mid \exists\, O \in \mathcal{O},\ x \in O \subseteq Y \}$

(1) $X \in \mathcal{N}(X, x)$

$\Rightarrow \mathcal{N}(X, x) \neq \phi$

(2) $\forall N \in \mathcal{V}(X, x)$, $x \in N$

(3) closed under finite intersection. If $N \subseteq U \Rightarrow U \in \mathcal{V}(X, x)$.

(4). $\forall N \in \mathcal{V}(X, x)$.

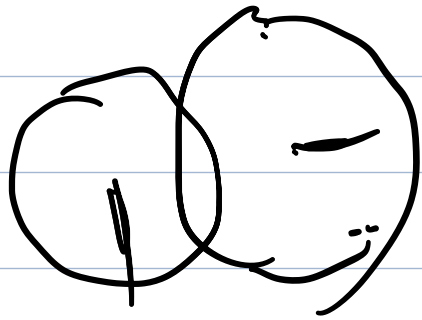
$$x \in O \subseteq N$$

$$\Rightarrow O \subseteq \mathring{N}, \quad \mathring{N} \in \mathcal{V}(X, x) \checkmark.$$

Topology basis

Define $\mathcal{B} = \{ B(x, r) \subseteq \mathbb{R}^n = X \}$

then $\forall B_1, B_2 \in \mathcal{B}$



Key property:

(1) $x \in B_1 \cap B_2$, $\exists B_3$, s.t.:

$x \in B_3 \subseteq B_1 \cap B_2$.

(2) $X = \bigcup_{B_i \in \mathcal{B}} B_i$

· $\mathcal{B} \Rightarrow \mathcal{N}$

$\mathcal{N}(X, x) := \{ N \mid \exists B \in \mathcal{B}, x \in B \subseteq N \}$

(1) $X \in \mathcal{N}(X, x)$

(2) $N \in \mathcal{N}(X, x) \Rightarrow x \in N$

(3) $N_1, N_2 \in \mathcal{N}(X, x)$

$\Rightarrow B_1, B_2 \in \mathcal{B}$

$\exists B_3 \subseteq B_1 \cap B_2$

$\Rightarrow N_1 \cap N_2 \in \mathcal{N}(X, x)$

(4) $\forall N \in \mathcal{N}(x)$

$N \subseteq M$

$\Rightarrow M \in \mathcal{N}(x)$.

(5). . . . .

$\mathcal{B} \Rightarrow \mathcal{N} \Rightarrow \mathcal{O}$

$O \in \mathcal{O} \Leftrightarrow \exists B, B \subseteq \mathcal{O}$

Closed sets.

$\mathcal{O} \to \mathcal{C}$

$\forall C \in \mathcal{C}(X)$ complement of open.

Subspace topology

$\mathcal{N}(X,x) \Rightarrow \mathcal{N}(Y,x)$.

$N \in \mathcal{N}(Y,x)$

$\Leftrightarrow \exists N' \in \mathcal{N}(X,x), N' \cap N(Y,x) = N$

$$\mathcal{O}(X) \Rightarrow \mathcal{O}(Y)$$

$$O \in \mathcal{O}(Y) \Leftrightarrow \exists O' \in \mathcal{O}(X),$$

$$O' \cap Y = O$$

---

f continuous

$\Leftrightarrow$ $\forall y \in Y$, $N \in \mathcal{N}(Y, y)$

$f^{-1}(N) \in \mathcal{N}(X, f^{-1}(y))$

$\Leftrightarrow$ $\forall O \in \mathcal{O}_Y$, $f^{-1}(O) \in \mathcal{O}_X$.

Thm. $f : X \to Y$ continuous

$A \subseteq X$

then $f|_A : A \to Y$ continous.

Thm. If $f : X \to Y$

$Y \subseteq Z$ subspace

then $X \to Y$ continuous

$\Leftrightarrow$ $X \to Y \xrightarrow{i} Z$ continuous

$\Rightarrow$: trivial.

$\Leftarrow$: $\forall O \in \mathcal{O}_Y$, $\exists O' \in \mathcal{O}_Z$, $O' \cap Y = O$

$\Rightarrow f^{-1}(O) = f^{-1}(O')$ open.

Thm. the following statements are equivalent.

(a) $f: X \to Y$ is continuous

(b) $\forall O$, $f^{-1}(O) \in \mathcal{O}_X$

(c) $\forall B \in \mathcal{B}(Y)$, $f^{-1}(B) \in \mathcal{O}(X)$

(d) $f(\overline{A}) \subseteq \overline{f(A)}$

(e) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

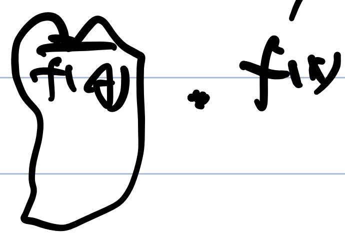
(f) $\forall C \in \mathcal{G}(Y)$, $f^{-1}(C) \in \mathcal{G}(X)$

(b) $\Rightarrow$ (d): $\forall A \subseteq X$

if $\exists x \in \overline{A}$, $f(x) \notin \overline{f(A)}$

$\Rightarrow \exists O \in \mathcal{O}(Y)$, s.t. $f(x) \in O$, $O \cap \overline{f(A)} = \emptyset$

$x \in f^{-1}(O) \in \mathcal{O}(X)$



But $f^{-1}(O) \cap A = \emptyset$

$\Rightarrow x \notin \overline{A}$, contradiction!

(d) $\Rightarrow$ (e).

$\forall B \subseteq Y$

$A = f^{-1}(B) \Rightarrow f(A) \subseteq B$

$\Rightarrow f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B}$

$\Rightarrow \overline{A} \subseteq \overline{B}$

$\Rightarrow \quad f^{-1}(B) \subseteq B$

(e) $\Rightarrow$ (f)

$\forall C \in G(Y)$

$\bar{C} = C$

$\overline{f^{-1}(C)} \subseteq f^{-1}(\bar{C}) = f^{-1}(C)$

$\Rightarrow f^{-1}(C) \in G(X)$

Example

$X = [0, 1)$, $Y = S^1$

$X \to Y$

$f : x \to e^{2\pi x i}$

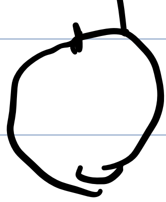
- f is continuous
- f is bijective
- f^{-1} is Not continuous

Def. $f: X \to Y$ is homeomorphism

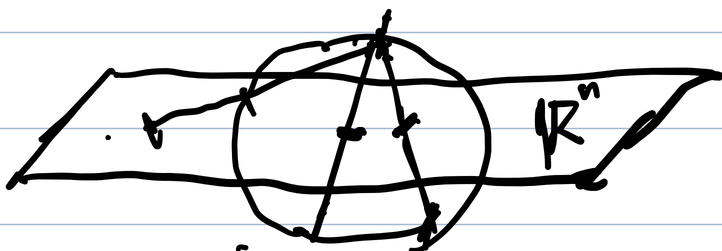
$\Leftrightarrow$ f bij, f, f^{-1} continuous

Example.

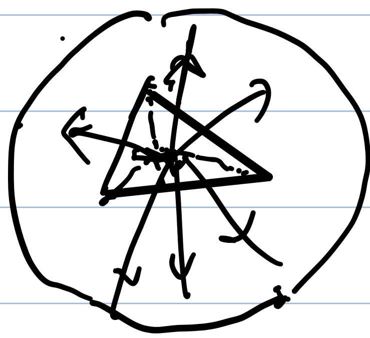
$p = (1, 0, \cdots, 0)$.



$f: S^n - \{p\} \to \mathbb{R}^n$.



f is homeomorphism.



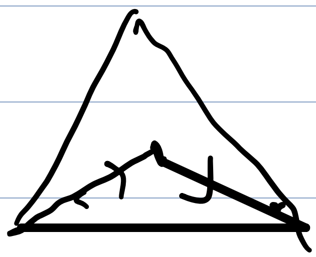
homeomorphism.

Space filling curve.

$[0,1) \to [0,1) \times [0,1)$

Construction of Peano's curve

$f_1 : [0,1] \to \Delta$



f_2

f_3

Convergence uniformly

- $|f_n(x) - f_{n+1}(x)| \le \frac{1}{2^{n-1}}$
- Let $f = \lim_{n\to\infty} f_n(x) \Rightarrow f$ continuous

$f : [0,1] \to \Delta$

- for all $p \in \Delta$, $\exists N, t$, s.t.

$$|f_N(t) - p| \leq \frac{1}{2^{N-1}}$$

hence $f([0,1])$ is dense in Δ

- $f([0,1])$ is compact

$$\Rightarrow f([0,1]) = \Delta$$

Prop. every metric space is Hausdorff

Lemma. $\forall A \subseteq X$, X metric space

define $d(x, A) = \inf\limits_{y \in A} d(x, y)$

Then $x \mapsto d(x, A)$ is continuous.

pf: $\forall x, y \in X$ $\forall z \in A$

$$d(x,z) - d(y,z) \le d(x,y)$$

$$\Rightarrow d(x,A) \le d(x,z) \le d(y,z) + d(x,y)$$

$$\Rightarrow d(x,A) - d(y,A) \le d(x,y)$$

Similarly, $|d(x,A) - d(y,A)| \le d(x,y)$.

$\square$

Lemma. X metric space

$A, B \in G(X)$, $A \cap B = \phi$

$\Rightarrow \exists f : X \to \mathbb{R}$ continuous. s.t.

$f(x) = 1$ on A, $f(x) = -1$ on B

and $-1 < f(x) < 1$ on $X - A - B$

pf: $f(x) = \dfrac{d(x,B) - d(x,A)}{d(x,B) + d(x,A)}$

Thm. Tietze extension ~~theorem~~.

If X is metric space

$A \in G(X)$, $f: A \to B$ continuous

$\Rightarrow f$ can extension to $g: X \to B$

$$g\big|_A = f$$

① Assume f is bounded.

$\exists M$, s.t. $\forall x \in A$, $|f(x)| < M$

$A_1 = f^{-1}([\frac{M}{2}, +\infty))$

...

See on textbook p 40

$\square$

$$
\begin{tikzcd}
 & & X \\
Z \arrow[r, "f"] \arrow[urr, bend left, "\pi_X \circ f"] \arrow[drr, bend right, "\pi_Y \circ f"'] & X \times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
 & & Y
\end{tikzcd}
$$

π_X, π_Y are continuous, open map

Theorem.

f is continuous

$\Leftrightarrow$ $\pi_X \circ f$, $\pi_Y \circ f$ is continuous.

proof: $\Rightarrow$: trivial.

$\Leftarrow$:

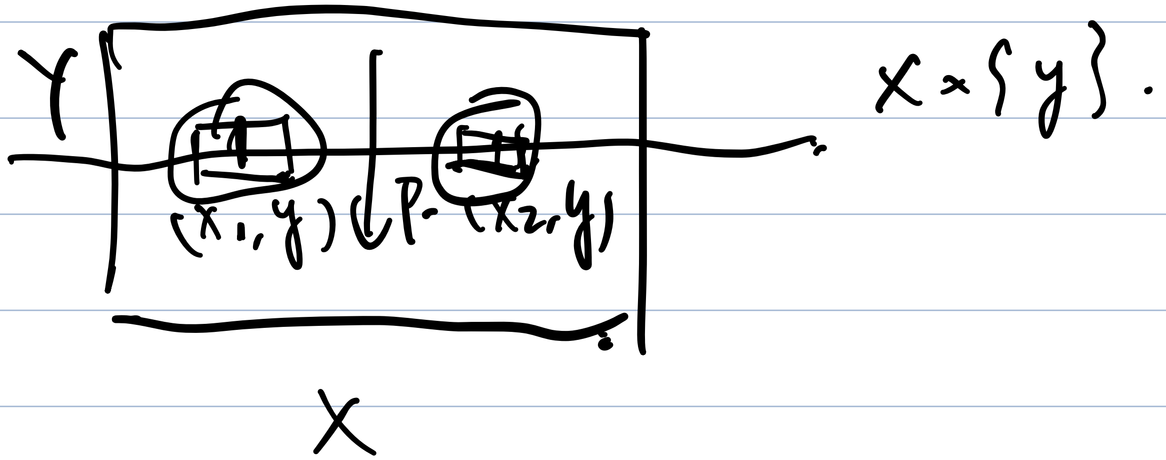
$$f^{-1}(U \times V) = (\pi_X \circ f)^{-1}(U) \cap (\pi_Y \circ f)^{-1}(V)$$

Theorem. $X \times Y$ hausdorff

$\Leftrightarrow$ X, Y are hausdorff.

$\Leftarrow$: trivial.

$\Rightarrow$:



Theorem. $X \times Y$ is cmpt

$\Leftrightarrow$ X and Y are cmpt.

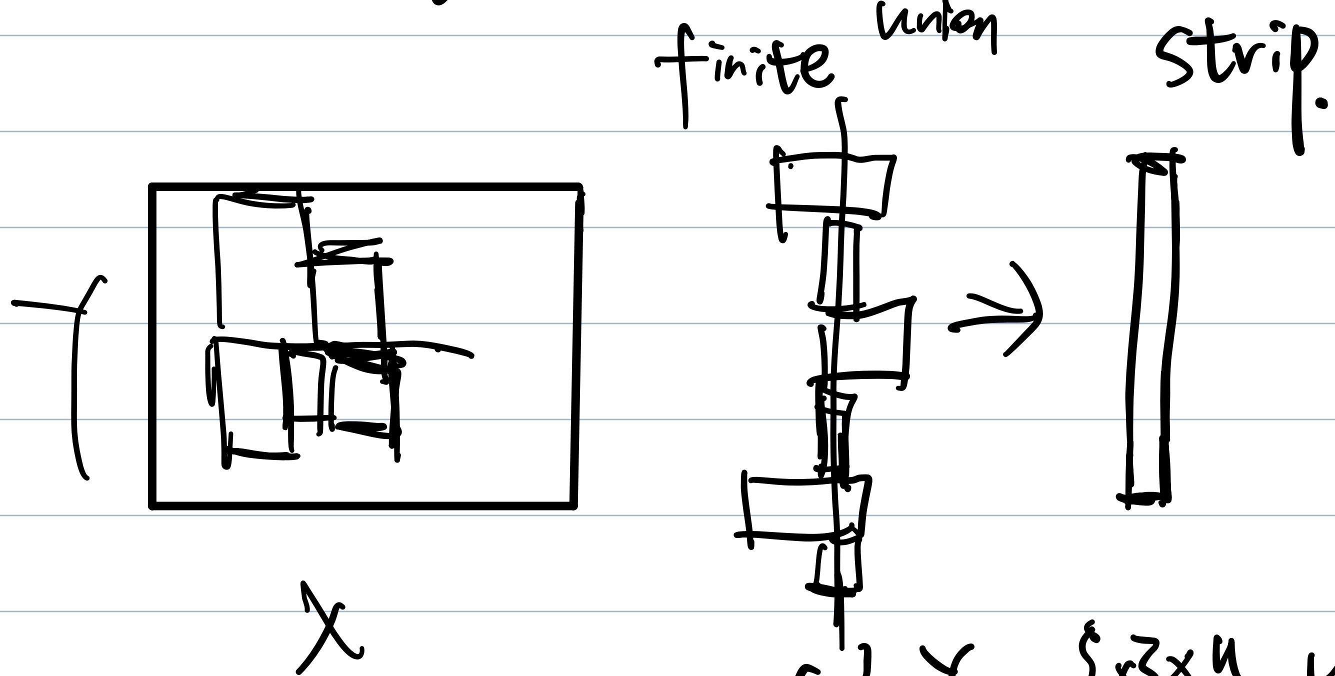
proof: if $X \times Y$ is cmpt.

$$\Rightarrow \quad \begin{aligned} X &= \pi_X(X\times Y) \\ Y &= \pi_Y(X\times Y) \end{aligned} \quad \text{are cmpt.}$$

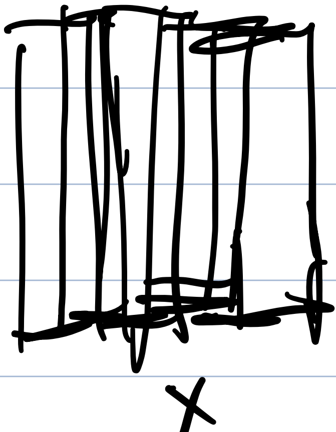
If X, Y are cmpt.

It's ~~enough~~ to check every cover of open basis has finite subcover.

$$X\times Y = \bigcup_{j\in J} U_j \times V_j$$



$\{x\}\times Y$ $\quad$ $\{x\}\times U$, U open

$\Rightarrow$  ✓.

finite union.

Y is cmpt

$\Rightarrow \{x\} \times Y$ is cmpt.

Connectness.

Def. X is connected

$\Leftrightarrow$ if $X = A \cup B$, $A, B \neq \phi$

$\Rightarrow \bar{A} \cap B \neq \phi$ or $A \cap \bar{B} \neq \phi$

Thm. TFSAE:

(1) X is connected

(2) $\mathcal{O}(X) \cap \mathcal{C}(X) = \{\phi, X\}$.

(3) X cannot be expressed as the disjoint union of two proper open sets.

(4) There is no onto map from X to a discrete space which has at least 2 points.

Thm. $X \subseteq \mathbb{R}$ is connected

($\Leftrightarrow$) X is an interval

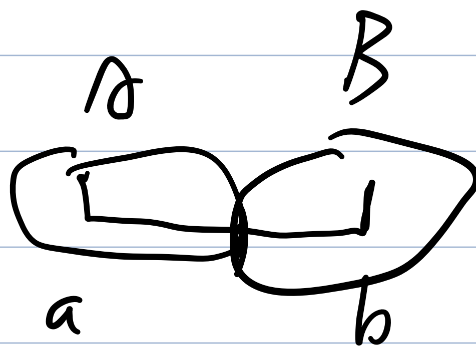
$\Rightarrow$: $(\inf X, \sup X) \subseteq X$

$\Leftarrow$: $X = A \cup B$, $A \cap B = \emptyset$,

$A, B \neq \emptyset$, $A, B \in \mathcal{O}(X)$

Take $a \in A$, $b \in B$

we can assume $a < b$



$\Rightarrow [a,b] \subseteq X$

Consider $A \cap [a,b] \neq \emptyset$

define $c := \sup(A \cap [a,b])$

$\Rightarrow c \in [a,b] \subseteq X$

$\Rightarrow c \in A$ (because A is closed)

$A \in \mathcal{O}(X)$ ball in X

$\Rightarrow \exists \varepsilon$, s.t. $B_\varepsilon(c) \subseteq A$

$\Rightarrow c = b$, contradiction!

Theorem.

If $f: X \to Y$ is continuous

X is connected

$\Rightarrow f(X)$ is connected

Pf: If $A \subseteq f(X)$

$A \in \mathcal{O}(f(X))$ and $A \in C(f(X))$

$\Rightarrow f^{-1}(A) \in \mathcal{O}(X), C(X)$

$\Rightarrow f^{-1}(A) = \phi$ or X

$\Rightarrow A = \phi$ or $f(X)$

Corollary: $X \xrightarrow{\text{homeomorphism}} Y$

X connected $\Leftrightarrow$ Y connected

Theorem. if X topology space

$Z \subseteq X$

If Z is connected and is dense in

X

$\Rightarrow X$ is connected

pf: X is disconnected

$X = A \cup B,\ A, B \neq \phi,\ A \cap B = \phi,$

$A, B \in \mathcal{O}(X)$

If Z is dense

$A \cap Z,\ B \cap Z \neq \phi$

$\Rightarrow Z$ is disconnected.

Claim: Z is dense

$\Leftrightarrow$ $\forall$ non-empty open set. A

$A \cap Z \neq \phi$

$\Rightarrow$: if $\exists A$ open, $A \cap Z = \phi$, $A \neq \phi$

$\Rightarrow \overline{Z} \subseteq X \setminus A$

$\Leftarrow$: $\checkmark$.

•

Definition.

$A, B \subseteq X$

Then A, B are separated from each other

$\Leftrightarrow \bar{A} \cap \bar{B} = \phi$

Theorem, If $X_i \subseteq X$, $i \in I$ are connected

if $\forall i, j \in I$ $\bar{X}_i \cap \bar{X}_j \neq \phi$, $X = \bigcup_{i \in I} X_i$

$\Rightarrow$ X is connected.

Remark. "glueing" some connected space

pf: If $A \in \mathcal{O}(X)$ and $A \in C(X)$

If $A \neq \phi$

$\Rightarrow \exists x \in A \subseteq X = \bigcup_{i \in I} X_i$

$\Rightarrow \exists i$, $x \in X_i \cap A \in \mathcal{O}(X_i) \cap C(X_i)$

$\Rightarrow \quad A \cap X_i = X_i$

$\Rightarrow \quad \overline{X}_i \subseteq A$ (because $A \in C(X')$)

$\Rightarrow \forall j \in I_j$

$$A \cap X_j \in \mathcal{O}(X_j) \cap G(X_j)$$

Then either $A \cap X_j = \phi$ or

$$A \cap X_j = X_j$$

If $A \cap X_j = \phi \Rightarrow X_j \in A^c$

$\Rightarrow \overline{X}_j \in A^c \Rightarrow \overline{X}_i \cap \overline{X}_j = \phi$ X.

$\Rightarrow \quad A \cap X_j = X_j \;, \forall j$

$\Rightarrow \quad A = X$

Theorem. $X \times Y$ is connected

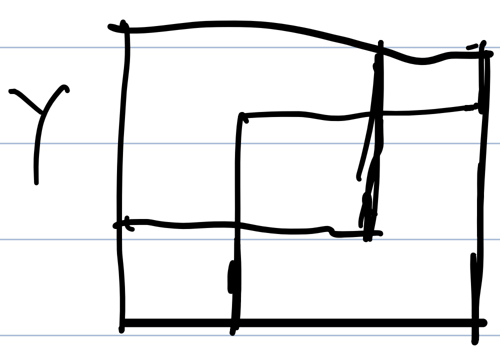
$\Leftrightarrow$ X, Y are both connected

Pf:

$\Rightarrow$: $X = P_x(X \times Y)$

$Y = P_Y(X \times Y)$

$\Leftarrow$:



x

$\forall x \in X, \ \forall y \in Y$

$A_{x,y} = (\{x\} \times Y) \cup (X \times \{y\})$

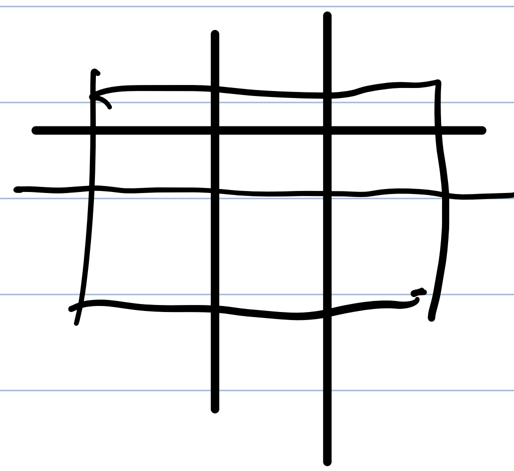
$\{x\} \times Y, \ X \times \{y\}$ is connected

$\overline{\{x\} \times Y} \cap \overline{X \times \{y\}} \neq \phi$

$\Rightarrow A_{x,y}$ is connected

Now consider

$$X = \bigcup_{\substack{x \in X \\ y \in Y}} A_{x,y}$$



$$\overline{A_{x_i,y_i}} \cap \overline{A_{x_j,y_j}} \supseteq \{(x_i,y_j),(x_j,y_i)\}.$$

$\Rightarrow$ X is connected.

$\square$

If X is a topological space, then we call $A \subseteq X$ as a connected components if A is connected, and if $A \subseteq B$, B connected $\Leftrightarrow$ $A=B$

Theorem. $X = \bigcup\limits_{\substack{A \text{ is a} \\ \text{connected} \\ \text{component}}} A$

And $A \cap B = \phi$, $\bar{A} = A$

$\forall A, B$ are connected component.

Pf: $\forall a \in X$

define $A := \bigcup_{\substack{a \in B \\ B \text{ is connected}}} B$

then A is connected (glueing B).

$x \in C$, C connected

$\Rightarrow C \subseteq A$

If A is a connected component

$\Rightarrow$ A is dense in $\bar{A}$

$\Rightarrow$ $\bar{A}$ is connected

If A, B are connected components

$A \cap B \neq \phi$

$\Rightarrow$ $A \cup B$ connected

$\Rightarrow$ $A = B$

Proposition. If $C \subseteq X$ connected

$\Rightarrow$ $\exists$ a connected component A

$C \subseteq A$

Let $A = \bigcup_{C \subseteq B} B.$

B is connected component

Example.

(1) X is connected

$\Rightarrow$ all of connected component are $\{X\}$.

(2) $\mathbb{R} \setminus \{-1, 1\}$ has 3 connected

component.

(3) $X = \mathbb{Q}$ with matric topology

$\Rightarrow$ every pt is a connected components connected subspace of $\mathbb{Q}$

$\Updownarrow$

$\sim$ of $\mathbb{R}$

If there are finitely many connected components

$\Rightarrow$ every connected components is open.

NOT true for $\mathbb{Q}$

Definition.

$\gamma : [0,1] \to X$ is called a path $\Leftrightarrow$ γ is continuous.

Definition.

X is called path-connected. if $\forall x, y$, $\exists$ path $\gamma : [0,1] \to X$

$\gamma(0) = x$, $\gamma(1) = y$

Thm. Path connected implies connected.

pf: If $X = A \cup B$, $A \cap B = \phi$,

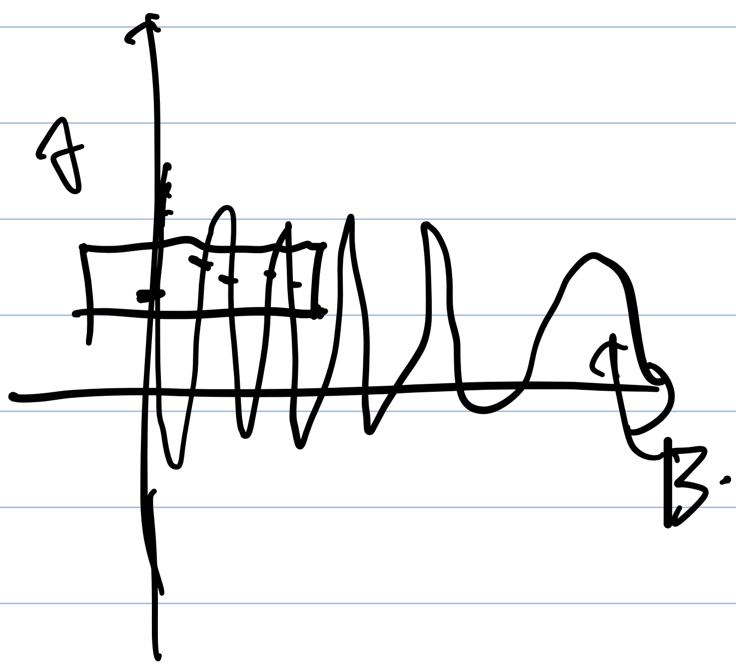
$A, B \in \mathcal{O}(X) \cap C(X)$, $A, B \neq \phi$

choose $x \in A$, $y \in B$.

$\exists\ \gamma$. $\gamma(0) = x$ $\gamma(1) = y$

Consider $\gamma^{-1}(A)$, $\gamma^{-1}(B)$.

Example.



Topologist's sine curve.

$Z = \{ (x, \sin\frac{1}{x}) \mid x \in (0,1] \}$.

$Y = \{ (0, y) \mid y \in [-1, 1] \}$.

$X = Z \cup Y$ $\quad$ Z is connected

Z dense in X

$\Downarrow$

X connected

X is not path connected:

$x = (0,1)$ $\quad$ $y = (1, 0)$

If $\exists\ \gamma$, s.t. $\gamma(0) = x$, $\gamma(1) = y$

Define $t_0 = \sup(\{ t : t_0, \gamma(s) \in Y, \forall s \in [0,t] \})$

Y is closed

$\Rightarrow \forall s \in [0, t_0]$. $\gamma(s) \in Y$

Since γ is continuous.

choose $\varepsilon << 1$, s.t.

$B(\gamma(t_0), \varepsilon) \subseteq I \times J$, $[-1,1] \subsetneq [b,c]$

$\exists \delta > 0$, s.t. $\gamma([t_0, t_0+\delta]) \subseteq I \times J$.

$\Rightarrow P_x \circ \gamma([0,1])$ is connected !

Theorem. If $A \subseteq \mathbb{R}^n$

A is open

then A connected

$\Updownarrow$

A is path connected

$\Leftarrow$: Proved.

$\Rightarrow$: $\forall x \in A$, define

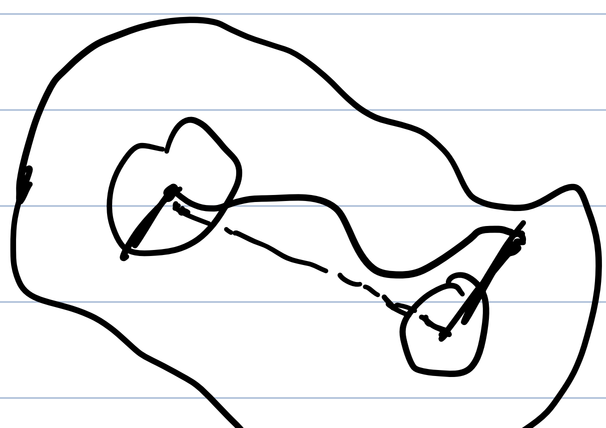
$A = \{ y \in A, \exists \gamma. \text{ s.t. } \gamma(0) = x, \gamma(1) = y \}$

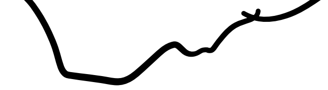
$B = A^c$.

Clearly $A \neq \phi$

Notice that A, B are both open.

$\Rightarrow B = \phi$





Equivalent relation.

Definition. If X is a Set. then $S \subseteq X \times X$ is called an equivalent relation if

$$x \sim y \iff (x, y) \in S$$

- $x \sim x$
- $x \sim y \iff y \sim x$
- $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Example: homeomorphism.

Example: If $\sim$ is an equivalent relation

$1 \sim 2$

$2 \sim 4$

$3 \sim 6$

$4 \sim ?$

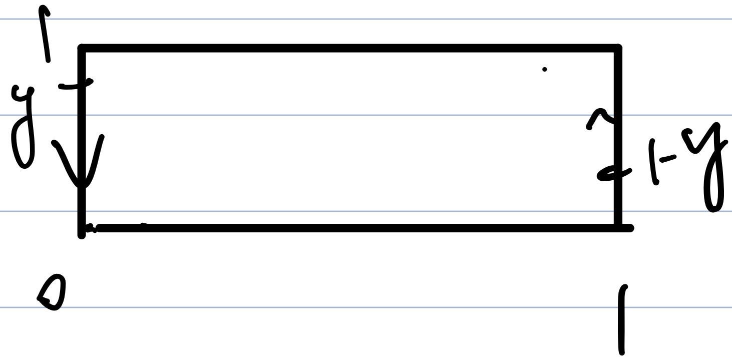
$\{1, 2, 4\} \subseteq ?$

$$X = \bigsqcup_{i \in I} X_i$$

分割.

is called a partition

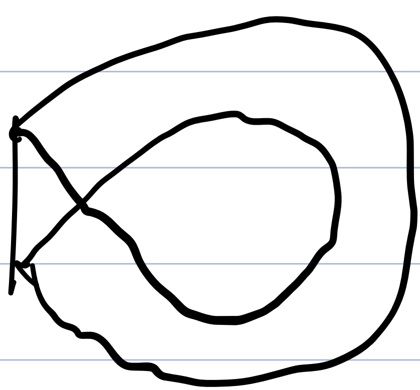
partition $\longleftrightarrow$ equivalent relation.



$$X = \bigcup_{(x,y)\in(0,1)\times[0,1]} \{(x,y)\} \cup \bigcup_{(x,y)\in\{0,1\}\times[0,1]} \{(0,y),(1,1-y)\}$$

X = Mobius strip.

$$= \left([0,1]^2\right) \Big/ \left((0,y)\sim(1,1-y)\right)$$



Definition. If

$X \to M$ is onto.

Then we define the quotient topology

$\mathcal{O}(M)$ on M by $O \in \mathcal{O}(M)$

$\Leftrightarrow \pi^{-1}(O) \in \mathcal{O}(X)$.

π is called an identification map.

quotient topology is the largest topology s.t. π is continuous.

coinduced topology

Universal property.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

g is continuous

$\iff g \circ \pi = f$ is continuous

pf: $\Rightarrow$: trivial

$\Leftarrow$: Conversely,

$\pi^{-1}(g^{-1}(\mathcal{O})) \in \mathcal{O}(X)$

$\iff g^{-1}(\mathcal{O}) \in \mathcal{O}(Y)$.

Remark.

Not any onto map is identification map.

Theorem.

$f: X \to Y$ onto map.

f is an identification map $\Longleftarrow$ f maps closed sets to closed sets.

Example.

B^n / S^{n-1}

$B^n = \{ |x| \le 1, x \in \mathbb{R}^n \}$

$$B^n / S^{n-1} = B^n \Big/ \left(\begin{array}{c} x, y \in S^{n-1} \\ \Rightarrow x \sim y \end{array} \right)$$

Claim: $B^n / S^{n-1} = S^n$.

$$\begin{tikzcd} B^n \arrow[r, "f"] \arrow[dr] & S^n \\ & B^n/S^{n-1} \end{tikzcd}$$

we need to construct an identify map f, s.t. its equivlent classes is $\{x\}$, $x \notin S^{n-1}$ and S^{n-1}

$f: B^n \to S^n$

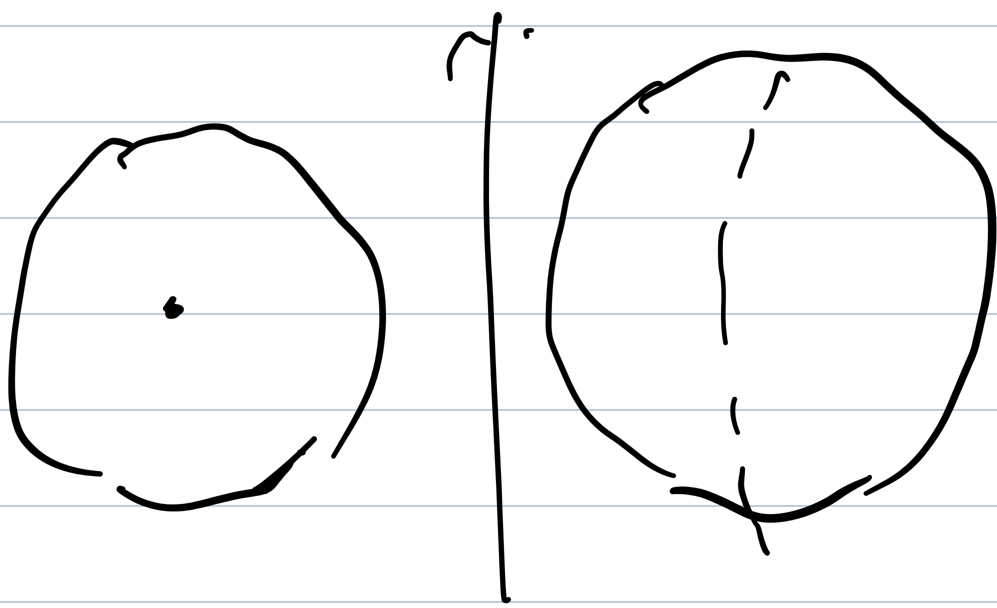
$$f(x) = \left(2|x|-1, \sqrt{1-(2|x|-1)^2}\,\frac{x}{|x|}\right) \in S^n \subseteq \mathbb{R}^{n+1}$$

f is continuous, and onto

B^n compact, S^n hausdorff.

$\Rightarrow$ f is open

$\Rightarrow$ f is identification.



In general, if $A \subseteq X$

$$X/A = \left(\bigcup_{x \notin A} \{x\} \right) \cup A$$

If $f: X \to Z$

$g: Y \to Z$

$$f \cup g : X \cup Y \to Z$$

$$f \cup g = \begin{cases} f(x), & x \in X \\ g(x), & x \in Y \end{cases}$$

Glueing Lemma.

$X \cup Y$ is topology space

X, Y closed

If $f: X \to Z$

$g: Y \to Z$

$$f\big|_{X \cap Y} = g\big|_{X \cap Y}$$

$\Rightarrow$ $f \cup g$ is continuous.

Def. If X, Y is topology space

then $X + Y = (X \times \{0\}) \cup (Y \times \{1\})$

If $B(X)$, $B(Y)$ is basis

Then let $B(X+Y)$

$= \{ B_1 \times \{0\}, B_1 \in B(X) \} \cup \{ B_2 \times \{0\},$

$\mathcal{B}_2 + \mathcal{B}_1$ 的.

$\Rightarrow \forall\, O \in \mathcal{O}(X+Y)$

$\Leftrightarrow \; O \cap X \in \mathcal{O}(X)$

$O \cap Y \in \mathcal{O}(Y)$

Lemma. if $X \in G(X \cup Y)$, $Y \in G(X \cup Y)$

$X+Y \to X \cup Y$ is identification

map.

$$X+Y \xrightarrow{j} X \cup Y \to Z$$

proof: j is continuous and onto

if $U \subseteq G(X)$ $V \subseteq G(Y)$

$j(u+v) = u \cup v$ is closed

$\Rightarrow j$ is closed

In general, $X = \bigcup_{i \in I} X_i$

Then define the disjoint union.

$$\bigoplus_{i \in I} X_i := \bigcup_{i \in I} (X_i \times \{i\})$$

This is coproduct in category of topology spaces

$u \in \mathcal{O}(\oplus X_i)$

$\Leftrightarrow u \cap X_i \in \mathcal{O}(X_i)$

$\oplus f_i \nearrow \quad Z$

$\nearrow \cup f_i$

$i\in I$ $\qquad\qquad$ $i \in I$

$$\bigsqcup_{i\in I} X_i \xrightarrow{\ \pi\ } \bigcup_{i\in I} X_i$$

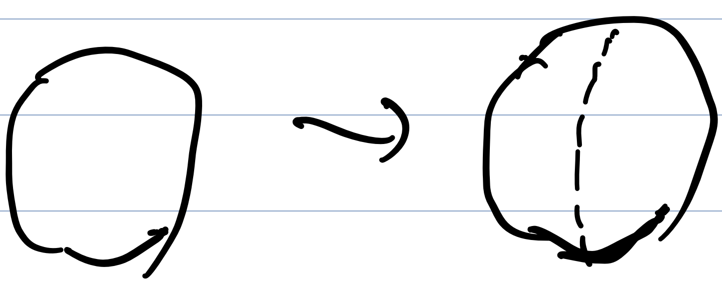
$\mathbb{RP}^n$

(a) $B^n / \{x \sim -x,\ |x| = 1\}$

(b) $S^n / \{x \sim -x\}$

(c) $\left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \mathbb{R}^*$

$B^n \to S^n.$



(a) $\rightsquigarrow$ (b).

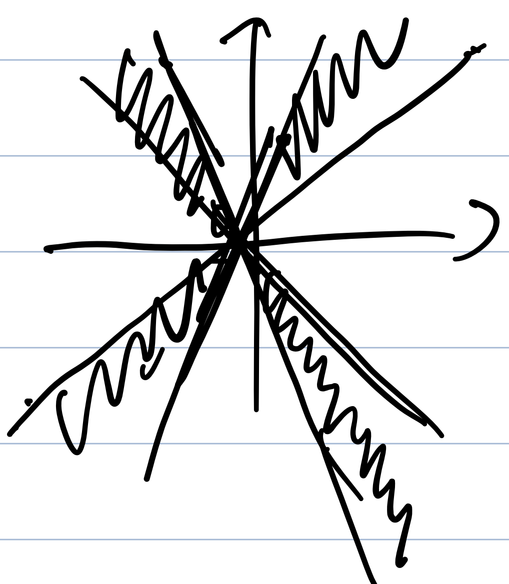
(2) $\Leftrightarrow$ (3):

$$S^n \to \mathbb{R}^{n+1} \to \mathbb{RP}^n$$

continuous, onto

$\mathbb{RP}^n$ is hausdorff, S^n is compact

$\Rightarrow$ this is identify map.



$\mathbb{CP}^n = S^{2n+1} / (x \sim y \Leftrightarrow \exists c \in S^1 \subseteq \mathbb{C}$, s.t. $x = cy)$

$$\mathbb{C}P^n = \left(\mathbb{C}^{n+1} - \{0\}\right) \Big/ \left(x \sim y \iff \exists c \in \mathbb{C}^*,\ x = cy\right)$$

$$\begin{tikzcd} S^{2n+1} \arrow[r] \arrow[dr, hook] & \mathbb{C}^{n+1} - \{0\} \arrow[r, "\sim"] & \mathbb{C}P^n \\ & S^{2n+1}/S^1 \arrow[ur] & \end{tikzcd}$$

$$\mathbb{C}P^n = \left.\underset{\text{closed ball}}{\overset{\mathbb{C}^n \atop \cup}{B^{2n}}}\right/ \left(x \sim y \iff \exists |z| = 1,\ \text{s.t. } x = zy \text{ and } |x| = |y| = 1\right)$$

$$= \underset{\text{open ball}}{B^{2n}} \cup \mathbb{C}P^{n-1}$$

$$B^{2n} \xrightarrow{\cdot} \mathbb{C}^n$$

$$B^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$$

$\forall x \in \mathbb{C}^n$

$(1, x) \in \mathbb{C}^n$

$$\mathbb{C}^{n+1} \xrightarrow{\pi} \mathbb{CP}^n$$

$\pi(1, x) \in \mathbb{CP}^n$

$\forall y \in \mathbb{C}^n - \{0\}, \ (0, y) \in \mathbb{C}^{n+1}$

$\pi(0, y) \in \mathbb{CP}^n$

$\forall c \neq 0 \quad \pi(0, y) = \pi(0, cy)$

$x \longmapsto \pi(0,x)$ / $y \to \pi(1,y)$

$$\begin{tikzcd}
\mathbb{C}^n - \{0\} \arrow[rr] \arrow[dr] & & \mathbb{CP}^n \\
& \mathbb{CP}^{n-1} \arrow[ur] &
\end{tikzcd}$$

- $\mathbb{C}\mathbb{P}^1 \simeq S^2$

To see this, we define the Hopf fibration map

$$(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2$$

$$f(z_1, z_2) = \left(|z_1|^2 - |z_2|^2,\ 2\mathrm{Re}(z_1 \overline{z_2}),\ 2\mathrm{Im}(z_1 \overline{z_2})\right) \in \mathbb{R}^3$$

$$(|z_1|^2 - |z_2|^2)^2 + 4\mathrm{Re}(z_1 \overline{z_2})^2 + 4\mathrm{Im}(z_1 z_2)^2$$

$$= 1$$

$\Rightarrow f : S^3 \to S^2$ continuous

$\forall y, z$ s.t. $y^2+z^2 = |z_1 \bar{z}_2|^2$

$\Rightarrow$ $\exists \theta$, s.t. $e^{i\theta} z_1 \bar{z}_2 = y + iz$

replace z_1 by $e^{i\theta} z_1$

$\Rightarrow$ f onto

$\Rightarrow$ f is identify map

$f(z_1, z_2) = f(z_1', z_2')$

$\Leftrightarrow \exists \theta, \quad (z_1, z_2) = e^{i\theta}(z_1', z_2')$

$\Rightarrow \mathbb{C}P \xrightarrow{\sim} S^2$

---

attaching map:

X, Y topological space.

$f(x)$

$\nearrow X$

$A \subseteq Y$ subspace, $A \hookrightarrow Y_A$

$f: A \to X$ continuous, injective

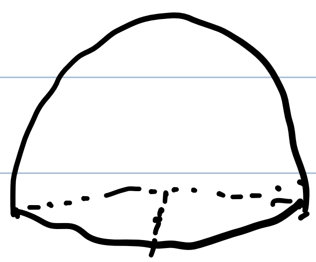
Then define $X \cup_f Y$ as

$$X \cup_f Y = \bigcup_{y \in A} \{y, f(y)\} \cup \left(\bigcup_{y \in Y \setminus A} \{y\} \right)$$

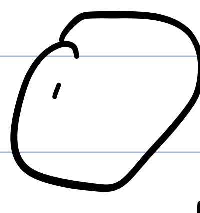
$$\cup \left(\bigcup_{x \in X \setminus f(A)} \{x\} \right)$$

with identification space

$$X + Y \to X \cup_f Y$$

$\mathbb{RP}^2 \cong$  $/ \sim : (x \sim y \iff x = -y)$

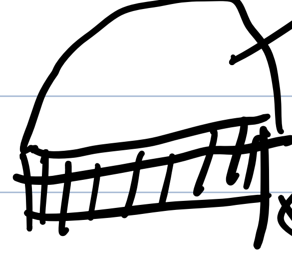
M

 $\xrightarrow{i}$  disc

boundary circle

disc

$P^2 = M \cup_i D$



Definition group.

Definition Topological group.

$m: G \times G \to G$

$(x, y) \to xy$ is continuous

and $i: G \to G$ is continuous.

$x \to x^{-1}$

$(S^1, \cdot) \subseteq \mathbb{C}^*$

Definition. G is a topological group

If $H \leq G$

H is topological subgroup.

(with subspace topology)

Example. Any group with discrete topology is a topological group.

Example.

$T^2 = S^1 \times S^1$

(Exercise. the product of topological group is topological group)

Definition.

$\forall (a,b,c,d), (e,f,g,h)$

$i^2 = j^2 = k^2 = -1$

$ij = k \quad jk = i \quad ki = j$

$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \setminus \{0\}$

with multiplicate at group ring.

define $\overline{a+bi+cj+dk} := a-bi-cj-dk$

$|a+bi+cj+dk| := \sqrt{a^2+b^2+c^2+d^2}$

$\Rightarrow x \cdot \overline{x} = |x|^2$

$\Rightarrow x^{-1} = |x|^{-2} \overline{x}$

$\Rightarrow$ $\mathbb{H}$ is closed under multiplicate

$GL_n(\mathbb{R})$

multiplicate, inverse can be viewed as rational function on $\mathbb{R}^{n\times n}$

Hence both continuous.

Similarly $GL_n(\mathbb{C})$

$SL_n(\mathbb{R})$ is topological subgroup of $GL_n(\mathbb{R})$

$O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I\}$

$x, y \in \mathbb{R}^{2n}$.

n

$SO(n)$. $\quad w(x,y) = \sum_{i=1} (x_i y_{n+i} - x_{n+i} y_i)$

$w(y,x) = -w(x,y)$

$Sp(2n,\mathbb{R}) = \{ A \in \mathbb{R}^{2n\times 2n},\ w(Ax,Ay) = w(x,y), \forall x,y \}$

$= \{ A : A^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

There are called classical groups.

Recall: Topological Manifold.

X is a hausdorff topological

space. $\forall x \in X$,

$\exists\ u_x$, s.t. $u_x \xrightarrow{\sim} \mathbb{R}^n$.

Hilbert's 5th. problem

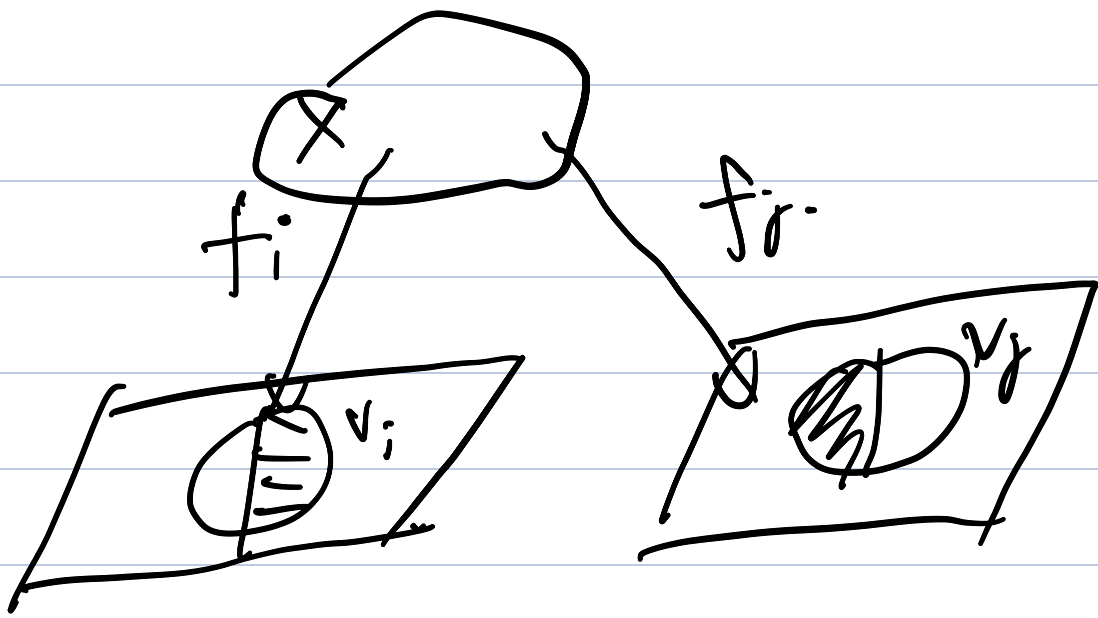
G is a topological manifold and a topological group

$\Rightarrow$ G is a Lie group.

Definition:

If X is a Hausdorff topological space $\exists\ X = \bigcup_{i \in I} u_i$, $f: u_i \longrightarrow V_i \subseteq \mathbb{R}^n$

homeomorphism.



$\forall\ i,j$ st. $u_i \cap u_j \neq \phi$

$$f_j \circ f_i^{-1} : f_i(u_i \cap u_j) \to f_j(u_i \cap u_j)$$

are C^∞, then call X a smooth manifolds.

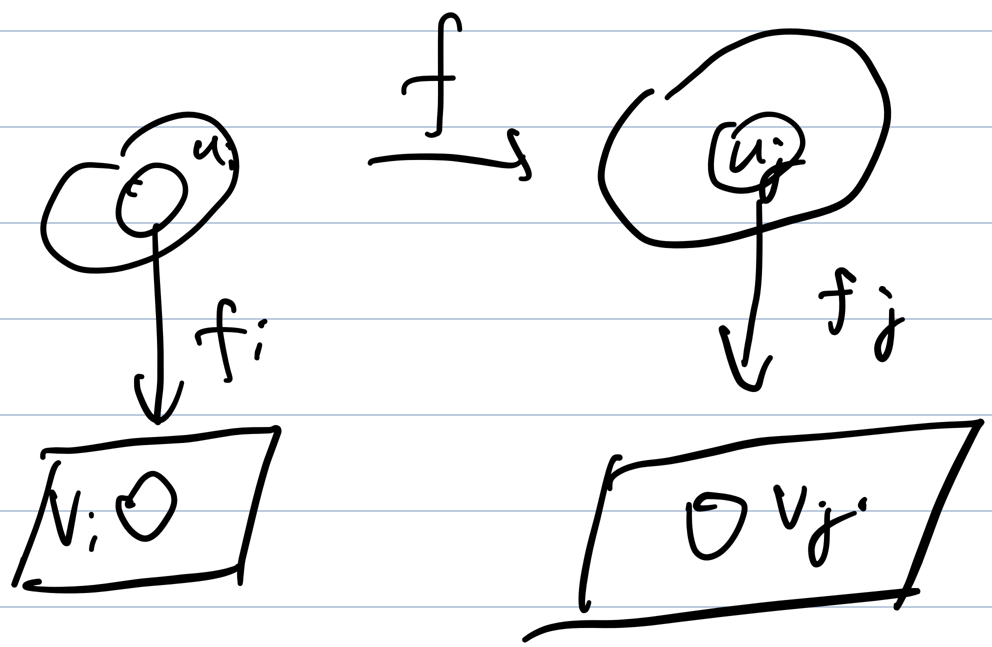
If X, Y are smooth manifolds

$f: X \longrightarrow Y$ is called a smooth

map $\Leftrightarrow$ $f_j \circ f \circ f_i^{-1}$:

$$f_i(u_i \cap f^{-1}(u_j)) \to f_j(f(u_i) \cap u_j)$$

is C^∞



Definition.

G is called a Lie group.

If G a smooth manifold and

$m : G \times G \to G$

$i : G \to G$

is smooth.

Classification of Lie group.

Definition.

If G, H are topological groups

then $f : G \to H$ is called

then $f: G \to H$ is called a

homomorphism, if:

(1) f is continuous

(2) f is group homomorphism.

f is isomorphism if it has a

Inverse.

If G is a tp gp

$\forall x \in G$, $L_x: G \to G$

$L_x(y) = x \cdot y$

$$G \to G \times G \to G$$

$$y \xrightarrow{i_x} (y, x) \xrightarrow{p} xy$$

$L_x = p \circ i_x \Rightarrow$ continuous.

$L_x^{-1} = L_{x^{-1}}$ is continuous

$\Rightarrow L_x$ is homeomorphism.

Thm.

If G is a topological group and K be the connected component which contains e, then

K is a closed normal subgroup

$\forall x \in K$

$K \cdot x^{-1} = \{ y \cdot x^{-1} : y \in K \}$

is a connected component which contains E

$\Rightarrow K \cdot x^{-1} = K$

Similarly, we have $a K a^{-1} = K$, $\forall a \in G$.

Thm. In a $\boxed{\text{connected}}$ topological group. G

$\forall u \in N(e, G)$, $G = \langle u \rangle$

Proof. $\forall x \in \langle u \rangle$

$L_x(u) \subseteq \langle u \rangle$,

$L_x(u) \in N(x, G)$

$\Rightarrow$ u is open.

$\forall x \notin \langle u \rangle$

$\forall y \in L_x(u)$

$\Rightarrow \exists z \in U,\ y = xz$

If $y \in \langle U \rangle \Rightarrow x \in \langle U \rangle$, $\times$

$\Rightarrow L_x(U) \subseteq G \setminus \langle U \rangle$

$\uparrow$

$N(x, U)$

$\Rightarrow G \setminus \langle U \rangle$ is open

By the connecteness,

$\langle U \rangle = G$

Compactness.

$O(n) = \{ A \in \mathbb{R}^{n\times n} \mid A^T A = I \}$

compact (bounded and closed).

$SO(n)$ compact, $\downarrow$.

$Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n)$ $\rightarrow$ unitary.

closed compact.

$\Rightarrow$ compact.

$U(1) = \{x \mid x\cdot\bar{x} = 1\} \xrightarrow{\sim} S^1 \leq \mathbb{C}\setminus\{0\}$.

$SU(1) = \{1\}$.

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} |a|^2+|c|^2=1 \\ |b|^2+|d|^2=1 \\ a\bar{b}+c\bar{d}=0 \\ \bar{a}b+\bar{c}d=0 \end{array} \quad ad-bc=1 \right\}.$$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $\underbrace{|a|^2+|b|^2=1}$

$S^3 \subseteq \mathbb{H} \setminus \{0\} \xrightarrow{} $ Hamilton's group.

$f: \mathbb{H}\setminus\{0\} \to \mathbb{C}^{2\times 2}$

$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad k \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$f\big|_{S^3}$ invertible $\to SU(2)$.

f, f^{-1} are both continuous.

homeomorphism.

$f(x\cdot y) = f(x)\cdot f(y)$.

$SO(2) \xrightarrow{\sim} S^1$

$SU(2) \to SO(3)$.

- $\forall\, x \in S^3 \subseteq \mathbb{H} \setminus \{0\}$

  $\forall\, y \in \operatorname{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$

$z = x y \bar{x}$ $\qquad \bar{z} = x \bar{y} \bar{x}$

$= -x y \bar{x} = -z$

$\Rightarrow z \in \operatorname{Im}\mathbb{H}$

$|z| = |y|$

$y \to x y \bar{x}$ linear

this map is in $O(3)$

Recall that $\forall A \in O(3)$

$\Rightarrow S^3 \to O(3) \xrightarrow{\det} \{1, -1\}$

By connectedness

$\Rightarrow S^3 \xrightarrow{f} SO(3)$ group homomorphism.

If $f(x) = 1$, then $\forall y \in \operatorname{Im} \mathbb{H}$

$x y \bar{x} = y$

$\Rightarrow xy = yx$ $\qquad x = a + bi + cj + dk$

$\Rightarrow i, j, k = 0$

$\Rightarrow x = \pm 1$

$\ker f = \{\pm 1\}$

Consider $\forall x \in S^3$

$f(x)(i) = x i \bar{x}$

View S^3 as $SU(2)$

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} i & -ib \\ -i\bar{b} & -ia \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2 i - |b|^2 i & -2iab \\ -2\bar{a}\bar{b} i & -i(|b|^2 - |a|^2) \end{pmatrix}$$

$i \in \operatorname{Im} \mathbb{H}$

$f(x) \in SO(3)$

$f(x)(i) = (|a|^2 - |b|^2, \; 2\operatorname{Im}(ab), \; -2\operatorname{Re}(ab))$

This is onto.

$\forall A \in SO(3)$

$Ai \in S^2$

$\Rightarrow \exists x$ s.t.

$f(x) i = Ai$

$A^{-1} f(x)\, i = i$

$\Rightarrow A^{-1} f(x) \in SO(2) = S^1$

(restrict to $\mathbb{R}i \oplus \mathbb{R}j$)

$A^{-1} f(x) j = \cos\theta\, j + \sin\theta\, k$

$A^{-1} f(x) k = -\sin\theta\, j + \cos\theta\, k$

$f(\cos\theta' + i\sin\theta')(j)$

$= \cos 2\theta'\, j + \sin 2\theta'\, k$

$f(\cos\theta' + i\sin\theta') k$

$= -\sin 2\theta'\, j + \cos 2\theta'\, k$

If $\theta = 2\theta'$, $A^{-1} f(x) = f(\cos\theta' + i\sin\theta')$

$\Rightarrow A \in \operatorname{Im} f$

$f : S^3 \to SO(3)$ f onto, continuous

S^3 compact, $SO(3)$ Hausdorff

$\Rightarrow f$ is identify map

$\ker f = \{\pm 1\}$ $\mathbb{RP}^3 \xrightarrow{\sim} SO(3)$

$\forall (x, y) \in S^3 \times S^3$ $S^3 \subseteq \mathbb{H}^3 \setminus \{0\}$

$x \mapsto xz\bar{y} \in \mathbb{H}$ defines a map

$$S^3 \times S^3 \to O(4) \xrightarrow{\det} \{1,-1\}$$

$$\Rightarrow S^3 \times S^3 \xrightarrow{g} SO(4)$$

$\ker g = \{(1,1),(-1,-1)\}$.

g onto (similar to previous proof)

$$(S^3 \times S^3)/\pm$$

* The key of proofs above is group action.

$S^3 \circlearrowleft$: , consider its stabilizer

SO_2.

$$\begin{tikzcd}
S^3 \times S^3 \arrow[d, "\ker = \{\pm(1,1)\}"'] \arrow[dr, bend left, "\ker = \{(\pm 1, \pm 1)\}"] & \\
SO(4) \arrow[r, dashed] & SO(3) \times SO(3)
\end{tikzcd}$$

self dual form.

$SO(n) \circlearrowright S^{n-1} \to S^{n-1}$

Definition, G is a topological group, X is a topological space, X is an continuous map

$$G \times X \xrightarrow{f} X$$

is group action, and continuous.

Remark.

$$X \to G \times X \xrightarrow{f} X$$
$$x \xrightarrow{i_x} (g, x) \to gx$$
$$\xrightarrow{\quad f_g \quad}$$

$\Rightarrow$ f_g is homeomorphism.

Definition.

A group action $G \curvearrowright X$ is called transitive if $\forall x, y$, $\exists g$, s.t. $g(x) = y$

$O(n) \curvearrowright S^{n-1}$ is transitive

Proof. $x, y \in S^{n-1}$

$|x| = 1$ $\exists$ othogonal basis

$e_1, \cdots, e_n$ $e_1 = x$

and $e_1', \cdots, e_n'$ $e_1' = y$

$\sum a_i e_i \xrightarrow{\varphi} \sum a_i e_i'$

$\Rightarrow \varphi \in O(n)$

change $\pm$ ($e_i \to -e_i'$)

$\Rightarrow \varphi \in SO(n)$.

Definition.

$G \curvearrowright X \quad \forall x \in X$

Define the isotropy subgroup

$G_x = \{ g \in G \mid g \cdot x = x \}$

If $\{x\}$ is closed in X

e.g. X is Hausdorff $\nearrow$ imply

$\Rightarrow$ G_x is closed

$G \to G \times X \to X$

$g \mapsto (g, x) \xrightarrow[\text{projection.}]{} x$

Proposition.

If $\exists g \in G$, s.t. $y = gx$

$\Rightarrow$ $G_x = g^{-1} G_y g$

$hx = x$

$\Leftrightarrow h g^{-1} y = g^{-1} y$

$\Leftrightarrow g h g^{-1} \in G_y \quad \Leftrightarrow h \in g^{-1} G_y \, g$

If $G \curvearrowright X$ is transitive

H is a subgroup of G,

$H \curvearrowright X$ is transitive, and

$H \supseteq G_x$, then

$H = G$

Pf: $\forall g \in G$

$gx \in X \Rightarrow \exists h$, s.t.

$gx = hx$

$\Rightarrow h^{-1}g \in G_x \in H \Rightarrow g \in H$

Example: $G \curvearrowright G$

By L_x, $R_{x^{-1}}$

$H \leq G$ $\quad H \curvearrowright G$ by L_x, $R_{x^{-1}}$, $L_x \circ R_{x^{-1}}$

$\mathbb{Z} \curvearrowright \mathbb{R}$

$(n, r) = n + r.$

Definition. $G \curvearrowright X$

$x \sim y \iff x, y$ are in a same orbit.

The quotient space

$X/\sim$ is called X/G

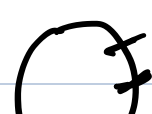
Example. $\mathbb{Z} \curvearrowright \mathbb{R}$

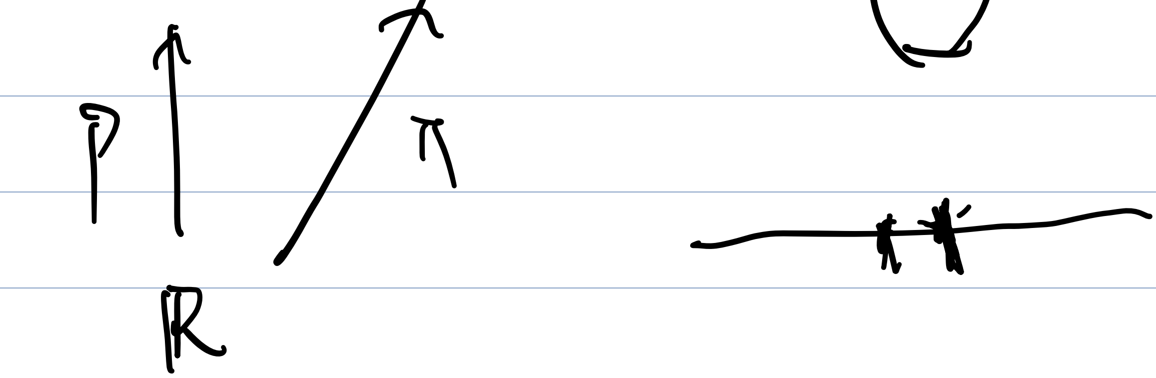
$\mathbb{R}/\mathbb{Z}$

Define $\mathbb{R} \to S^1 \subseteq \mathbb{C}\setminus\{0\}$.

$r \mapsto e^{2\pi r i}$ continuous, onto.

$\mathbb{R}/\mathbb{Z} \dashrightarrow S^1$





$\forall$ open set $O \subseteq \mathbb{R}$

$\pi(O)$ is open

Assume that isometry.

$$G \le \mathrm{Iso}(\mathbb{R}^n) = O(2) \ltimes \mathbb{R}^2$$

G is discrete

$\forall p \in \mathbb{R}^n$, $\forall r > 0$

$\{g \in G, |g(p) - p| < r\}$ is finite

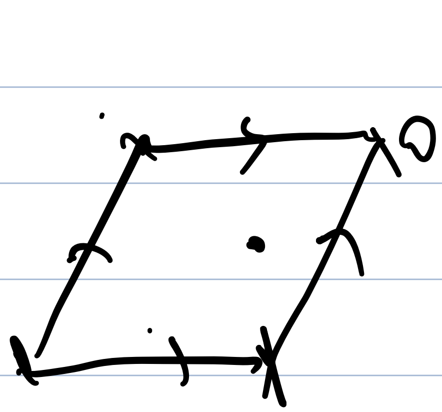
Then we define the fundamental domain.

$$D = \{x \in \mathbb{R}^n \mid \forall g \in G, |x-0| \le \underline{|x-g(0)|}\}$$

Then $D/\sim$ <u>is homeomorphic to</u>

$\underline{\mathbb{R}^n/G}$

$x \sim y \Leftrightarrow \exists g, x = gy$



$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "\pi_1"'] \arrow[dr, "F"] & \mathbb{R}^n \arrow[d, "\pi_2"] \\ D/\sim \arrow[r, "f"'] & \mathbb{R}^n/G \end{tikzcd}$$

$\forall\, O \in \mathcal{O}(D/\sim)$

$$f(O) = \bar{f}(\pi_1^{-1}(O))$$

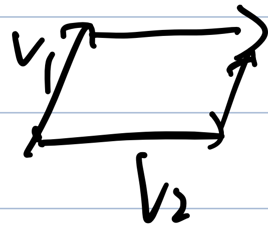
$$= \pi_2(i(\pi_1^{-1}(O)))$$

It's equivalent to show

$\pi_2^{-1}(f(O))$ is open.

Example 1:

$$\left\langle \begin{matrix} x \to x+v_1 \\ x \to x+v_2 \end{matrix} \right\rangle$$

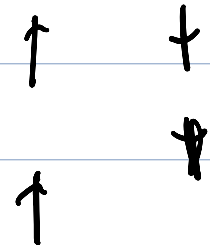


$\mathbb{R}^2/G$ = torus

Example 2.

$g(x,y) = (x+1, -y)$ $\qquad G = \langle g, h \rangle$

$h(x,y) = (x+1, 1-y)$



$\Rightarrow$ G is discrete

$h \circ g(x,y) = (x+2,\ y+1)$

$g \circ h(x,y) = (x+2,\ y-1)$

$g^2(x,y) = (x+2, y)$

$h^4(x,y) = (x-1, 1-y)$

$h^2 = (x+2, y) = g^2 = a$

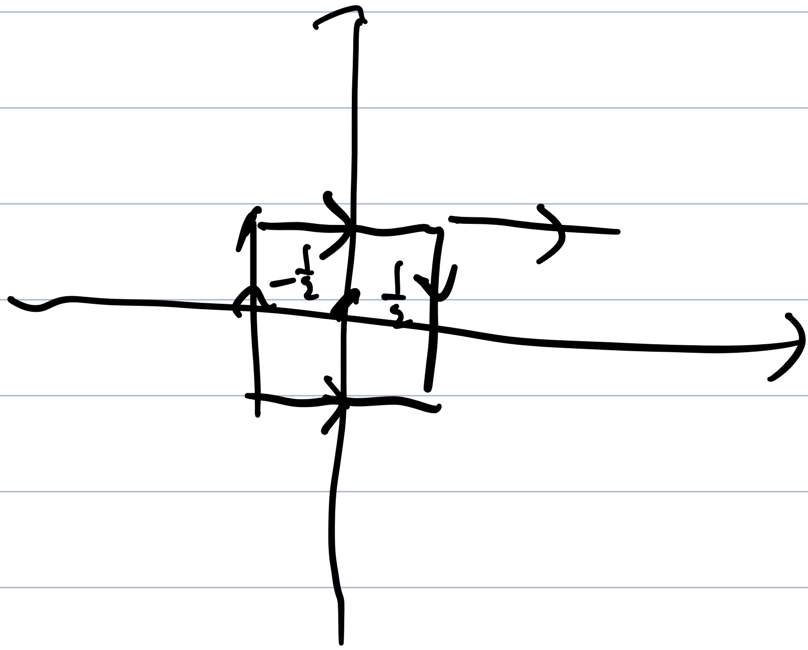
$b = g h g^{-1} h^4$

$= (x, y-2)$

Any elements of G has the

form: $a^{\square} b^{\square} h^{\square} g^{\square}$

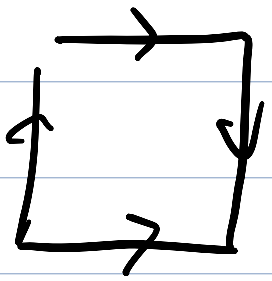
$ba = ab \cdot b^{-1} a^{-1} ba$



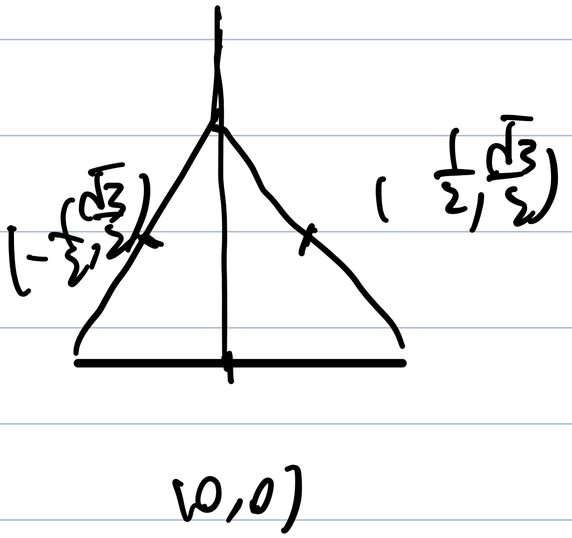
$O(O) = \mathbb{Z} \times \mathbb{Z}$

$D = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$

$h \quad f : g$

$\mathbb{R}^2/G =$  $=$ Klein Bottle.

Example 3.



$g(x,y) = (-x, -y)$

$h(x,y) = (1-x, \sqrt{3}-y)$

$f(x,y) = (-1-x, \sqrt{3}-y)$

$g^2 = h^2 = f^2 = e$

$$g^{-1}h^{-1}gh = ghgh$$

$$= (x-2, y-2\sqrt{3}) = a.$$

$$b = g^{-1}f^{-1}gf = (x+2, y-2\sqrt{3}).$$

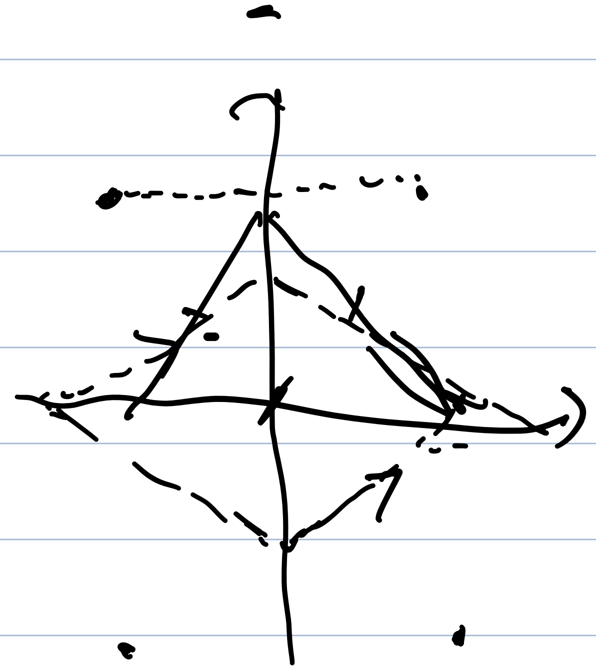
$$f^{\mathbb{Z}} g^{\mathbb{Z}} f^{\mathbb{Z}} h^{\mathbb{Z}} \cdots$$

$$gf = afg$$

$\Rightarrow$ Any element of G can be expressed

as $a^{\mathbb{Z}} b^{\mathbb{Z}} c^{\mathbb{Z}} f^{\mathbb{Z}} g^{\mathbb{Z}} h^{\mathbb{Z}}$

$$ab^{-1} = c.$$



$D/\sim \xrightarrow{\sim} \mathbb{RP}^2$

Hilbert's problem.

If G is discrete

$\forall p, r$ $\{g \in G, |g(p)-p| < r\}$ is finite

If $\mathbb{R}^n/G$ is compact

$\Rightarrow \exists H \leq G$, H is Abelian, normal,

G/H is finite, and

$G/H \leq C(n)$

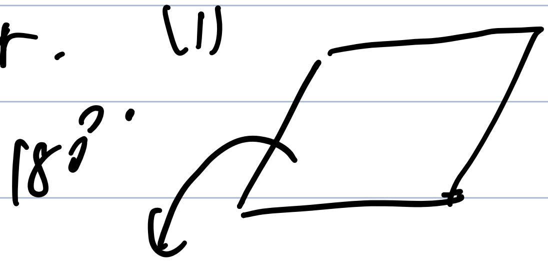
Definition. G is called a crystallographic group.

Example: If $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, $\tau \in \mathbb{C}$

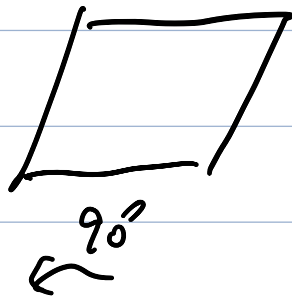
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{g} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Lambda & \xrightarrow{h} & \mathbb{C}/\Lambda \end{array}$$

When can we find $g \in \mathrm{Iso}(\mathbb{C})$, s.t. g can induce h?

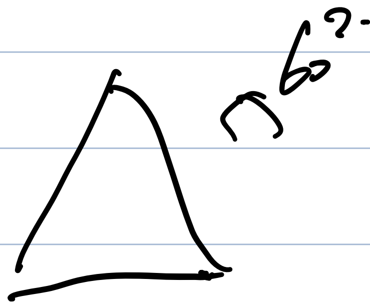
Answer. (1)



(2)



(3)



Pf:

$$g(a+b\tau)$$

$$= (1\ \tau)\begin{pmatrix} x_1 & \overset{0}{\overset{\|}{x_2}} \\ -x_3 & x_4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \quad A \in GL(2, \mathbb{Z})$$

assume $g(0)=0$, $\det g = 1$

$\Rightarrow g(z) = e^{i\theta} z$

$\Rightarrow (1 \ \tau) A = (1 \ \tau) e^{i\theta}$

$\Rightarrow \det(A - e^{i\theta} I) = 0$

Compute θ.

Proposition. If $G \curvearrowright X$ transitely, G is compact, X is Hausdorff, $G \curvearrowright X$

$\Rightarrow G_x \curvearrowright G$ by $R_{x^{-1}}$

The quotient space $G/G_x = X$

e.g.

$SO(n) \curvearrowright S^{n-1}$

$\boxed{G_x = SO(n-1).}$

proof: fix $x \in X$

We can define

$\varphi: G \to X$ $\qquad G \to G \times X \xrightarrow{m} X$

$g \mapsto gx$ $\qquad g \to (g, x)$

Continuous, onto

$\Rightarrow$ This is an identification map.

$\varphi(g) = \varphi(h)$

$\Leftrightarrow \quad g^{-1}h \in G_x$

$\Leftrightarrow h \in g G_x$

$\Leftrightarrow$ h, g is in the same orbit

of

$G_x \curvearrowright G$

$g \cdot h := h g^{-1}$

Lemma. If $G \curvearrowright X$ is a group action. X/G and G are connected

$\Rightarrow$ X is connected

Proof: If $X = U \cup V$, U, V are open,

$U \cap V = \phi$, $U, V \neq \phi$

$\Rightarrow$ (Exercise 2). $X \xrightarrow{\pi} X/G$ is open)

$\Rightarrow$ $\pi(U)$ and $\pi(V)$ are all open

$\pi(U) \cup \pi(V) = X/G$

式 $\pi(U) \cap \pi(V) \neq \phi$

$\exists x \in \pi(U) \cap \pi(V)$

$\Rightarrow \exists y \in X, \pi(y) = x$

$\Rightarrow \pi^{-1}(x) = G \cdot y =$ orbit of y

There is a continuous map

$f: G \to \pi^{-1}(x)$

$g \mapsto g \cdot y$

$\pi^{-1}(x) \cap U, \pi^{-1}(x) \cap V \neq \phi$

There are in $\mathcal{O}(\pi^{-1}(x))$

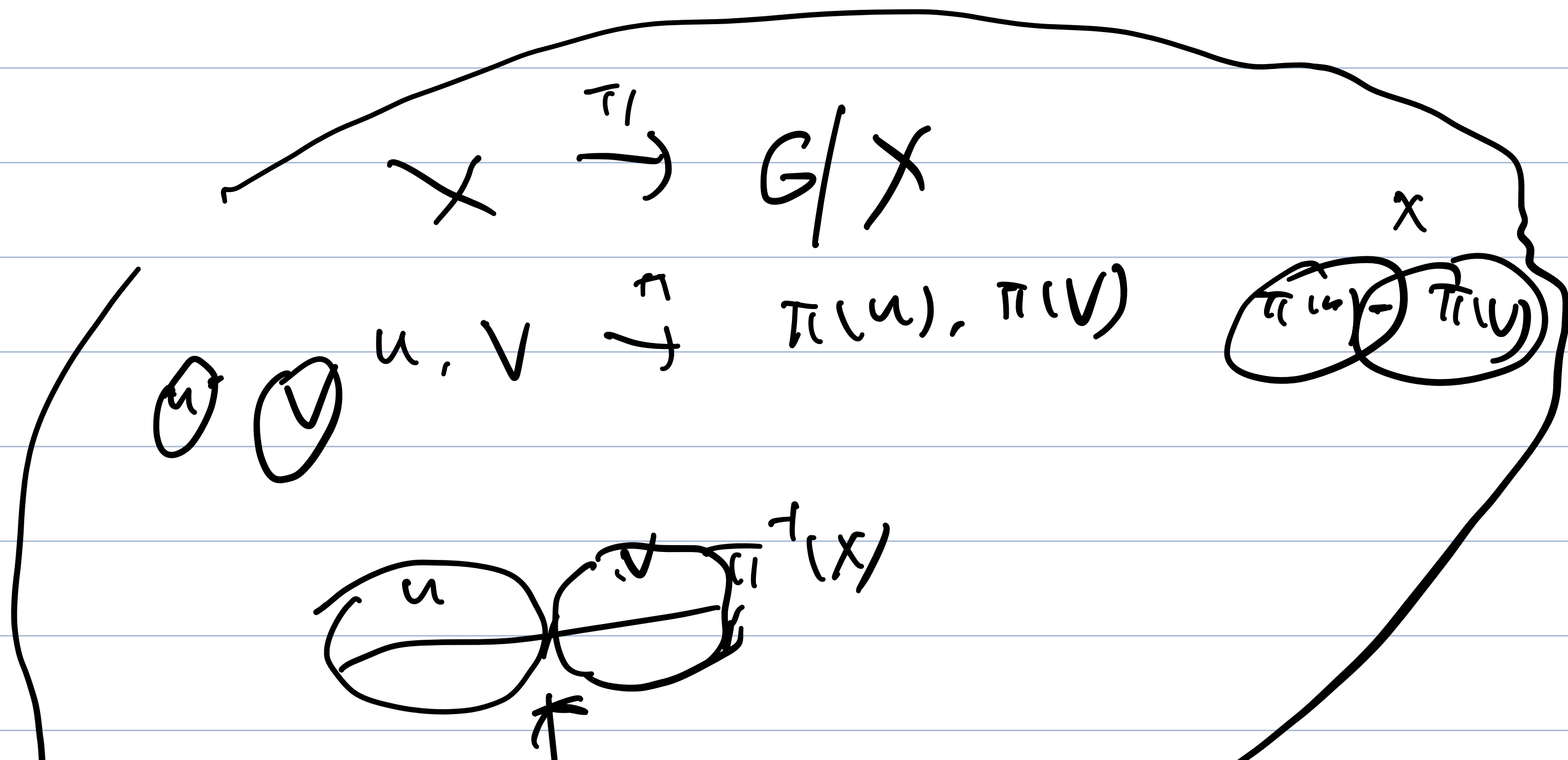
$\Rightarrow$ $f^{-1}(\pi^{-1}(x)\cap U)$, $f^{-1}(\pi^{-1}(x)\cap V)$

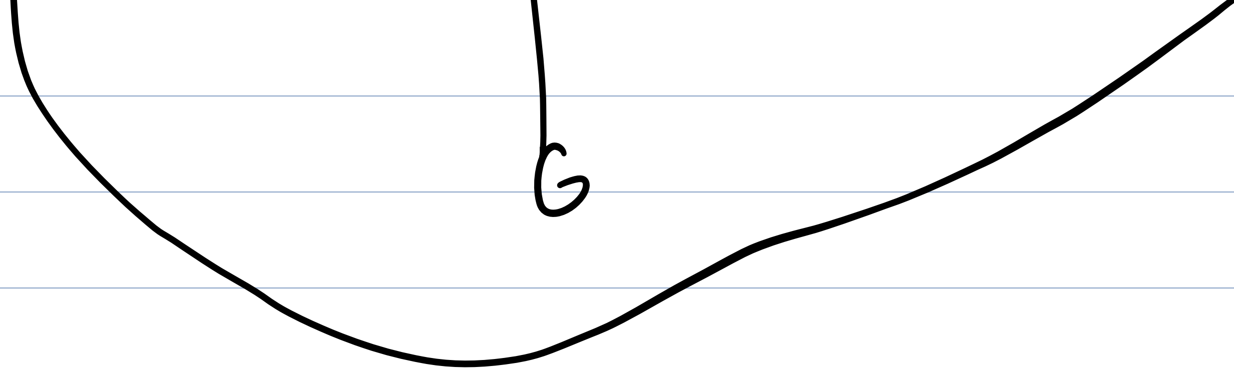
are open, non-empty in G

$$f^{-1}(\pi^{-1}(x)\cap U)\cup f^{-1}(\pi^{-1}(x)\cap V)=G$$

$$f^{-1}(\pi^{-1}(x)\cap U)\cap f^{-1}(\pi^{-1}(x)\cap V)=\phi$$

Condict to the connectness of G





$\Rightarrow$ $SO(n)$ is connected

$$SU(n)/SU(n-1) = S^{2n-1}$$

$$G_i \curvearrowright X_i$$

$$\Rightarrow G_1 \times \cdots \times G_n \curvearrowright X_1 \times \cdots \times X_n$$

$$\Rightarrow (X_1 \times \cdots \times X_n)/(G_1 \times \cdots \times G_n)$$

$$= (X_1/G_1) \times \cdots \times (X_n/G_n)$$

proof:

I will prove

$$X_1 \times \cdots \times X_n \xrightarrow{\pi} (X_1/G_1) \times \cdots \times (X_n/G_n)$$

is identification map

$$\pi(U_1 \times \cdots \times U_n)$$

$$= \underbrace{\pi_1(U_1)}_{\text{open}} \times \pi_2(U_2) \times \cdots \times \pi_n(U_n)$$

is open.

Example.

$$S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$$

$S^1 \curvearrowright S^3$

$e^{i\theta}(z_0, z_1) = (e^{i\theta} z_1, e^{i\theta} z_1) = S^2 = \mathbb{CP}^1$

$\mathbb{Z}_p = \langle g \mid g^p = 1 \rangle \curvearrowright S^3$ by

$g(z_0, z_1) = (e^{\frac{2\pi i}{p}} z_0, e^{2\pi i q/p} z_1)$

$S^3/\mathbb{Z}_p$ is called lens space

(透镜空间)

$L(p, q)$

$\mathrm{Iso}(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R}^n, |f(x) - f(y)| = |x-y|$

$\forall x, y \}$.

$$= \{ Ax+b,\ A \in O(n) \}$$

$$= O(n) \ltimes \mathbb{R}^n.$$

with product topology

Suppose $G \subseteq Iso(\mathbb{R}^n)$ is discrete,

$\Rightarrow \forall p \in \mathbb{R}^n, \forall r > 0,$

$\{ g \in G,\ |gp - p| < r \}$ is finite.

$$D = \{ x \in \mathbb{R} \mid s.t.\ |x - 0| \leq |x - g0| \}$$

Topology. mid, Problem 2.

$\mathbb{R}, \mathbb{C}, \mathbb{H}$

$$m: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$\mathbb{C} \times \mathbb{C} \to \mathbb{C}$$
$$\mathbb{H} \times \mathbb{H} \to \mathbb{C}$$

$\mathbb{R}$ bilinear.

Norm, $|\cdot|$

Theorem. As long as we have $\mathbb{R}$-bilinear multiplication, s.t.

$$|z \cdot w| = |z| \cdot |w|$$

Then it must be

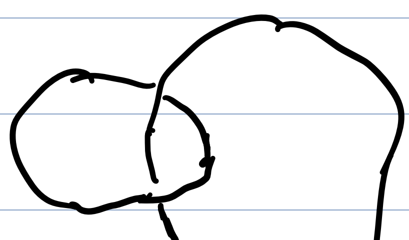
$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} = \mathbb{R}^8$$

$$\mathbb{O} = \bigoplus_{i=0}^{7} \mathbb{R} e_i$$

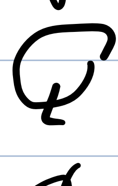
$$\Rightarrow \varphi_{ijk} = \left\langle \frac{1}{2}(e_i e_j - e_j e_i), e_k \right\rangle$$

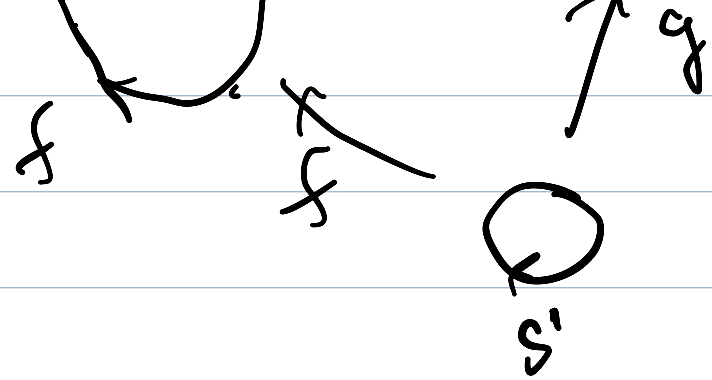
$$1 \le i, j, k \le 7$$

Homotopy.



g.





Def.

If $f, g : X \to Y$ are continuous.

Then we call f is homotopic to g, denotes by $f \sim g$.

$\iff \exists F : X \times [0,1] \to Y$ is continuous.

$$F(x,0) = f(x)$$

$$F(x,1) = g(x).$$

Def. if $A \subseteq X$

$f, g : X \to Y$ are continuous

then $f \simeq g$ rel A if

$\exists F : X \times [0,1] \to Y$ is continuous,

$F(x,0) = f(x)$, $F(x,1) = g(x)$

and $F(a,t) = f(a)$, $\forall a, t$.

i.e. $F(A,t)$ agrees on A.

Example.

if $C \subseteq \mathbb{R}^n$ is a convex subset i.e. $\forall x, y \in C$, $\lambda x + (1-\lambda) y \in C$, $\lambda \in [0,1]$

$\Rightarrow \forall f, g : X \to C$

let $F : X \times [0,1] \to C$

$$F(x,t) = t f(x) + (1-t) g(x)$$

$$\begin{tikzcd} X \times [0,1] \arrow[r] & \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \arrow[r] & \mathbb{R}^n \\ (x,t) \arrow[r] & (f(x), g(x), t) \arrow[r] & t f(x) + (1-t) g(x) \arrow[from=2-1, to=2-3, bend right, "F"'] \end{tikzcd}$$

$\Rightarrow F$ is continuous.

Proposition.

Homotopy is an equivalent relation.

Proof. $f \sim f$.

If $f \overset{F}{\sim} g$

$\exists F: X \times [0,1] \to Y$

$F(x,0) = f(x)$

$F(x,1) = g(x)$

$X \times [0,1] \to X \times [0,1] \to Y$

$$\begin{tikzcd}
(x,t) \arrow[r] \arrow[rr, bend right, "\bar{F}'"'] & (x,1-t) \arrow[r] & F(x,1-t)
\end{tikzcd}$$

$\bar{F}'$ is continuous. $\Rightarrow g \sim f$

If $f \overset{F}{\sim} g$ $\quad g \overset{G}{\sim} h$

$$H(x,t) = \begin{cases} F(x,2t), & t \le \frac{1}{2} \\ G(x,2t-1), & t \ge \frac{1}{2} \end{cases}$$

Glueing Lemma.

$\square$

If $f, g : X \to S^n$

If $\forall x \in X$, $f(x) \neq -g(x)$

$$F(x,t) = \frac{(1-t)f(x) + t g(x)}{|(1-t)f(x) + t g(x)|}$$

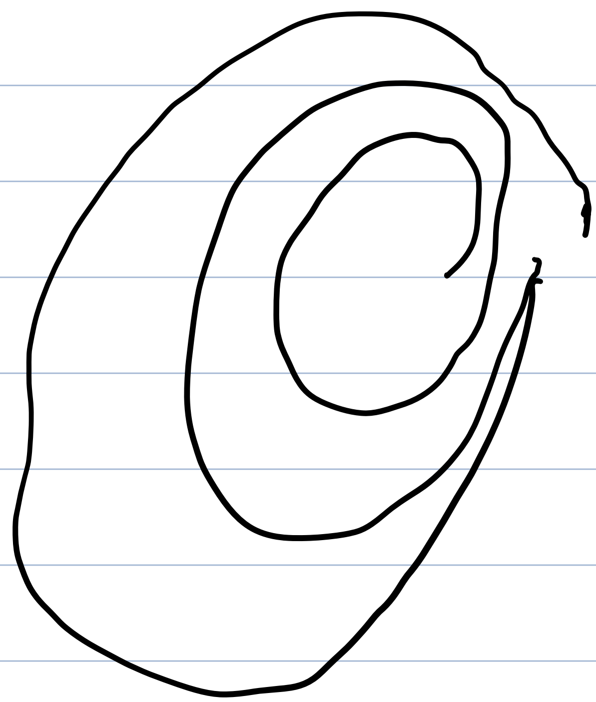
$$\begin{tikzcd} X \times [0,1] \arrow[r, "{(f,\,g,\,t)}"] \arrow[rr, bend right, "F"'] & S^n \times S^n \times [0,1] \arrow[r] & \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \arrow[d] \\ & & S^n \end{tikzcd}$$

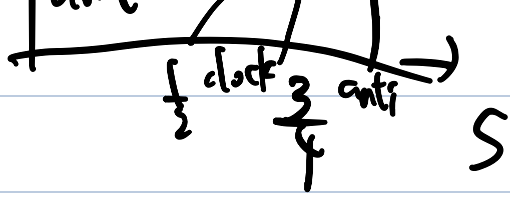
$\Rightarrow f \sim g$.

$f, g : [0,1] \to S^1$

$$f(s) = \begin{cases} e^{4\pi i s} & s \leq \frac{1}{2} \\ e^{4\pi i (2s-1)} & \frac{1}{2} \leq s \leq \frac{3}{4} \\ e^{8\pi i (1-s)} & \frac{3}{4} \leq s \leq 1 \end{cases}$$

$$g(s) = e^{2\pi i s}$$





Def. If X is a topological space, $p \in X$, then $\pi_1(p, X)$

$= \underbrace{\{ f : [0,1] \to X,\ f(0) = f(1) = p \}}_{:= \Omega(p).} / \sim$

If $f, g : [0,1] \to X$

Define multiplication as $f \cdot g : [0,1] \to X$

$$f \cdot g(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Apply glueing Lemma, we know

$f \cdot g$ is continuous

$$\begin{tikzcd}
\Omega(P) \times \Omega(P) \arrow[r] \arrow[d] & \Omega(P) \arrow[d] \\
\pi_1(P) \times \pi_1(P) \arrow[r, dashed] & \pi_1(P).
\end{tikzcd}$$

We want to obtain this, we

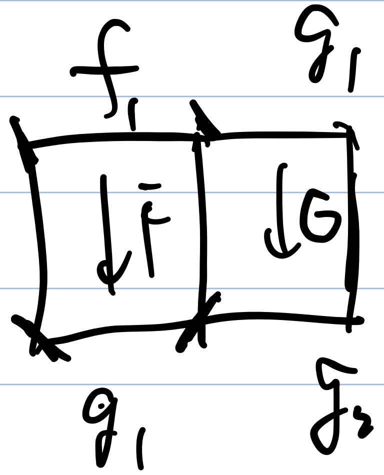
need to show if

$f_1 \overset{F}{\simeq} f_2 \text{ rel } \{0,1\}$, $g_1 \overset{G}{\simeq} g_2 \text{ rel } \{0,1\}$

Then Let.

$H: [0,1]\times[0,1] \to X$

$$H(s,t) = \begin{cases} F(s,2t) & t \le \frac{1}{2} \\ G(s,2t-1) & \frac{1}{2} \le t. \end{cases}$$



$g_1 \cdot f_1 \overset{H}{\sim} g_2 \cdot f_2$

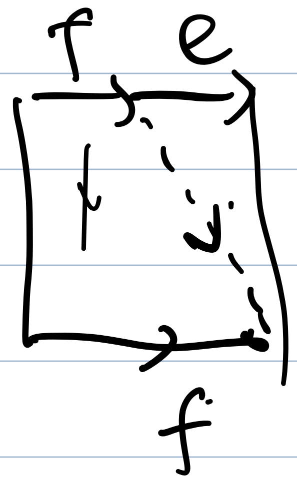
If $f: [0,1] \to X$ is continuous.

$f(0) = f(1) = p$

$f^{-1}(t) := f(1-t)$ is continuous.

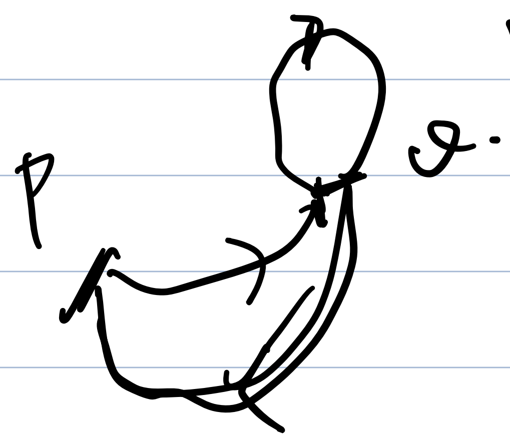
The operator $^{-1}$ is well defined.

$e(t) = p$. unit.



α p.

$(\gamma^{-1} \cdot \alpha) \cdot \gamma$



$\gamma(0) = p$

$\gamma(1) = q$.

$(\gamma^{-1} \cdot \alpha)(p)$

Recall. $\mathbb{Z} \curvearrowright \mathbb{R}$

$(n, x) \to n + x$

$\mathbb{R}/\mathbb{Z} = S^1$

Recall.

discrete subgroup $G \subseteq \mathrm{Iso}(\mathbb{R}^n)$.

$\forall p \in \mathbb{R}^n$

$\{g \mid |g(p) - p| < r\}$ is finite.

$\Rightarrow \mathbb{R}^n / G = \underbrace{D}_{\text{fundamental domain.}} / G$

Now we add a condition.

(No fix point)

$\forall p, \forall g \neq e, \; g(p) \neq p.$

Lemma. $G \curvearrowright \mathbb{R}^n$ is proper and has no fix point

$\Leftrightarrow \forall p \in \mathbb{R}^n, \exists u$ open, $p \in u$, s.t. $\forall g \neq e, \; u \cap g(u) = \phi$

Pf: $\Rightarrow$: $\forall p \in \mathbb{R}^n$

choose $g_0 \in G$

$r = |p - g_0 p|$

Then $\{g \in G \mid |g(p) - p| < r\}$ is

finite

$\Rightarrow \inf\limits_{g \in G \setminus \{0\}} |p - g(p)|$ can be achieved.

and it is > 0

Let $u = B(p, \frac{r}{2})$

$\Leftarrow$: No fix point is trivial.

u is open $\Rightarrow \exists \varepsilon > 0$.

$B(p, \varepsilon) \subseteq u$, $\forall r > 0$, $|g(p) - p| < r$.

$$\boxed{g_1(u) \cap g_2(u) = \phi \text{ for } g_1 \neq g_2}.$$

$\Rightarrow B(g(p), \varepsilon)$ are disjoint with each other.

$G \to G \cdot p$ (orbit) discrete

$g \to g(p)$ is bij. continuous

$\Rightarrow$ G is discrete

Definiton.

A map $\pi : \tilde{X} \to X$ is called a covering map if $\forall x \in X$, $\exists\, x \in V \subseteq X$, s.t.

$$\pi^{-1}(V) = \bigcup_{\alpha} U_\alpha, \quad U_\alpha \subseteq \tilde{X} \text{ are}$$

disjoint open sets, $\pi|_{U_\alpha} : U_\alpha \to V$ is homeomorphism.

If $G \curvearrowright X$ s.t. $\forall p \in X$, $\exists u$

open, $x \in u \subseteq G$

$$u \cap g(u) = \phi, \ \forall g \neq e.$$

Then $\pi : X \to X/G$ is a covering

map.

$\forall x \in X/G$, choose $y \in \pi^{-1}(x)$

find $y \in u \subseteq G$, $u \cap g(u) = \phi, \forall g \neq e$

$\Rightarrow \pi(u)$ is open, $x \in \pi(u)$

$$\pi^{-1}(\pi(u)) = \bigcup_{g\in G} g(u) \quad \text{pairwise disjoint}$$

$$\pi\big|_{g(u)} : g(u) \to \pi(u)$$

This is a bijective, continuous, open map, hence a homeomorphism.

$$t(x,y) = (x+1, y) \qquad G = \langle t, u \rangle$$

$$u(x,y) = (-x+1, y+1)$$

$$X = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}.$$

$PSL(2,\mathbb{R}) \curvearrowright X$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Theorem. If X is simply connected,

$G \curvearrowright X$, $\forall p$, $\exists\, p \in U \subseteq G$, $U \cap g(U) = \emptyset$

$\Rightarrow \pi_1(X/G) = G$

Lemma. Path-lifting lemma.

$\pi : \tilde{X} \to X$ is a covering

map,

$\forall\, q \in \tilde{X},\ p = \pi(q) \in X$

$\forall\, \gamma : [0,1] \to X$ continuous, $\gamma(0) = p$

$\Rightarrow \exists!\ \tilde{\gamma} : [0,1] \to \tilde{X}$ continuous,

$\pi \circ \tilde{\gamma} = \gamma,\ \tilde{\gamma}(0) = q.$

Proof. For all $x \in X$, $\exists$ open set

$x \in V_x \subseteq X$

$\gamma^{-1}(V_x) = \cdots$

$X = \bigcup_x V_x$

Lebesgue's Lemma, choose ε.

$\Rightarrow \exists\, t_0 = 0 < t_1 < \cdots < t_m = 1$, s.t.

$|t_i - t_{i+1}| < \varepsilon$,

$$[t_i, t_{i+1}] \subseteq \gamma^{-1}(V_{x_i})$$

$\gamma([t_0, t_1]) \subseteq V_{x_0}$

$$\pi^{-1}(V_{x_0}) = \bigcup_{\alpha} U_\alpha$$

$q \in U_{\alpha_0}$

$\pi|_{U_{\alpha_0}} : U_{\alpha_0} \to V_{x_0}$ is a homeomorphism

$\tilde{\gamma} : [t_0, t_1] \to \tilde{X}$ $\quad \tilde{\gamma} = (\pi_{u_{\alpha_0}})^{-1} \circ \gamma$

For $[t_1, t_2]$, repeat this. s.t.

$\tilde{\gamma}(t_1)$, $\tilde{\gamma}(t_0)$ is the same.

Glueing Lemma $\Rightarrow \tilde{\gamma}$

Uniqueness:

If $\tilde{\gamma}$, $\tilde{\gamma}'$, satisfies the same property

$$\Rightarrow \pi \circ \tilde{\gamma}'\big|_{[t_0,t_1]} : [t_0,t_1] \to V_{x_0}$$

$$\tilde{\gamma}'\big|_{[t_0,t_1]} : [t_0,t_1] \to \bigcup_{\alpha} U_\alpha$$

U_α are disjoint open sets.

Connectness of $[t_0,t_1]$

$$\Rightarrow \tilde{\gamma}'\big|_{[t_0,t_1]} : [t_0,t_1] \to U_{\alpha_1}$$

$\tilde{\gamma}'(0)=q \quad \Rightarrow \quad U_{\alpha_1} = U_{\alpha_0}$

$\square$

Lemma. Homotopy lifting Lemma.

If $\pi: \tilde{X} \to X$ is a covering map,

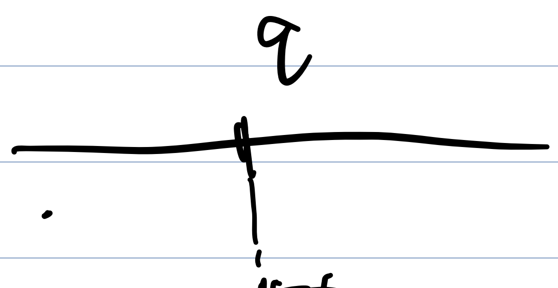
$\forall q \in \tilde{X}$, $\pi(q) = p$.

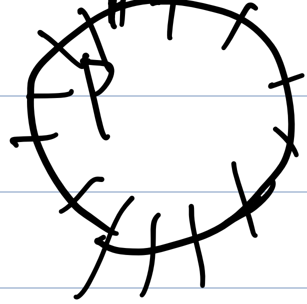
If $F: [0,1] \times [0,1] \to X$ is continuous,

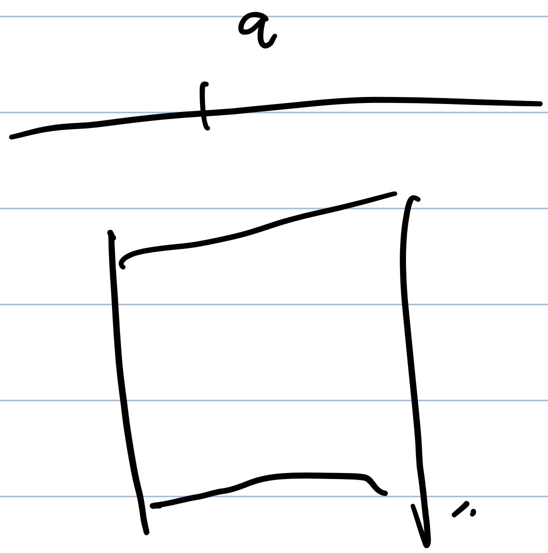
$F(0,t) = F(1,t) = p$. $\forall t$

$\exists! \ \tilde{F}: I \times I \to \tilde{X}$, s.t.

$\pi \circ \tilde{F} = F$, $\tilde{F}(0,t) = q$.







Proof of Main theorem.

$(\pi_1(X/G) = G)$

Proof. choose $q \in \pi^{-1}(p)$

$\forall g \in G$, X is path-connected

$\exists \gamma : [0,1] \to X$

$\gamma(0) = q$, $\gamma(1) = g(q)$

$\pi \circ \gamma : [0,1] \to X/G$ is a loop.

If γ' is another path

$\gamma' : [0,1] \to X$, $\gamma'(0) = q$,

$\gamma'(1) = g(q)$

$\Rightarrow$ $\gamma \cdot (\gamma')^{-1} \sim e_q$ rel $\{0,1\}$.

$\Rightarrow \langle \pi \circ \gamma \rangle = \langle \pi \circ \gamma' \rangle$

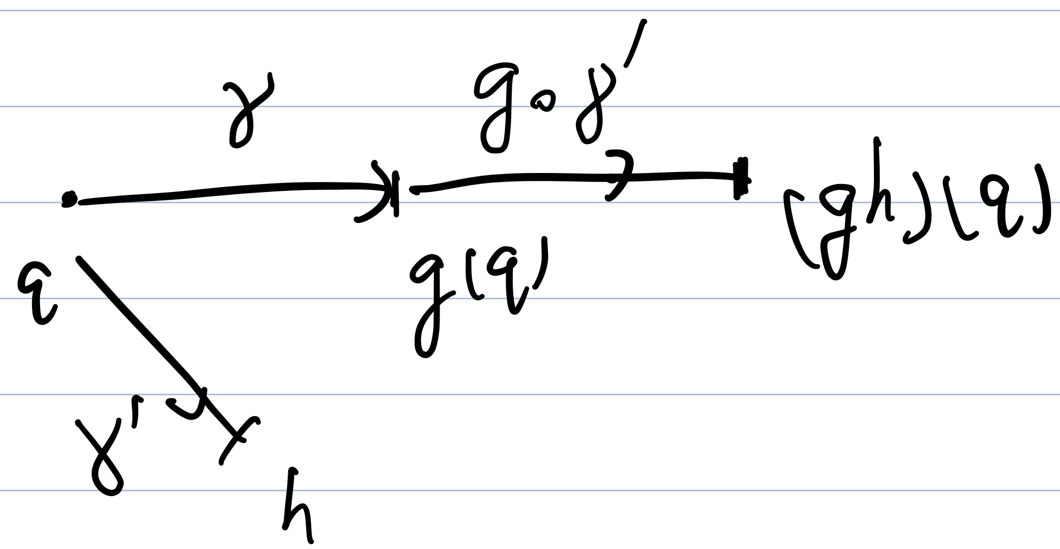
Then we can define

$$G \to \pi_1(p, X/G)$$

$g \to \langle \pi \circ \gamma \rangle$, γ begin at q

end at $g(q)$

If $g, h \in G$



$\phi(gh) = \langle (\pi \circ \gamma) . (\pi \circ (g \circ \gamma')) \rangle$

$= \langle \pi \circ \gamma \rangle . \langle \pi \circ \gamma' \rangle$

Path lifting

$\Rightarrow$ Surjective

If $g \in G$, $\phi(g) = \langle e \rangle$

Then $\pi \circ \gamma \overset{F}{\simeq} e$ rel $\{0,1\}$.

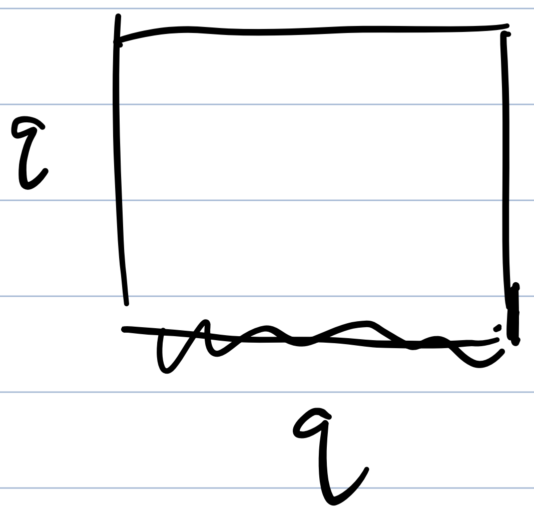
By homotopy lifting Lemma.

$\exists \tilde{F}$, $\pi \circ \tilde{F} = F$.

$\pi \circ \tilde{F}(1,t) = p$ $\quad \tilde{F}(0,t) = p$

uniqueness of path. $\Rightarrow$ $\tilde{F}(1,t) = q$

$\Rightarrow g = e$.



If $\pi : \tilde{X} \to X$

is a covering map

$\tilde{X}$ is path connected

$q \in \tilde{X}$ $\quad\quad$ $\pi(q) = p \in X$

$H = \pi_1(q, \tilde{X})$ $\quad\quad$ $G = \pi_1(p, X)$

Theorem.

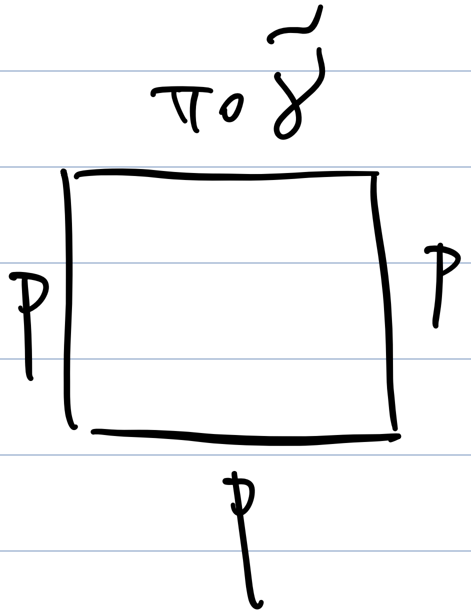
$\pi_*: H \to G$ $\quad$ is injective

Corollary. $H \xrightarrow{\sim} \pi_*(H)$

Proof. If $\langle \tilde{\gamma} \rangle \in \pi_1(q, \tilde{X})$ s.t.

$\langle \pi \circ \tilde{\gamma} \rangle = \langle e \rangle$

$\pi \circ \gamma \overset{F}{\simeq} e$

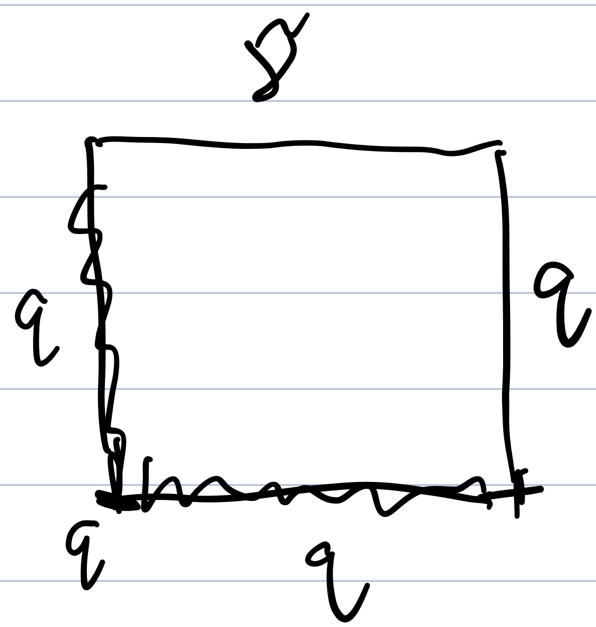


homotopy lifting Lemma

$\exists \tilde{F} : [0,1] \times [0,1] \to \tilde{X}$

s.t. $\pi \circ \tilde{F} = F$ and

$\tilde{F}(0,t) = q$



$\Rightarrow \langle \tilde{\gamma} \rangle = \langle e \rangle$

Theorem.

$\forall \alpha \in \Omega(p, X)$

$\exists \tilde{\alpha}(0) = q$, $\pi \circ \tilde{\alpha} = \alpha$

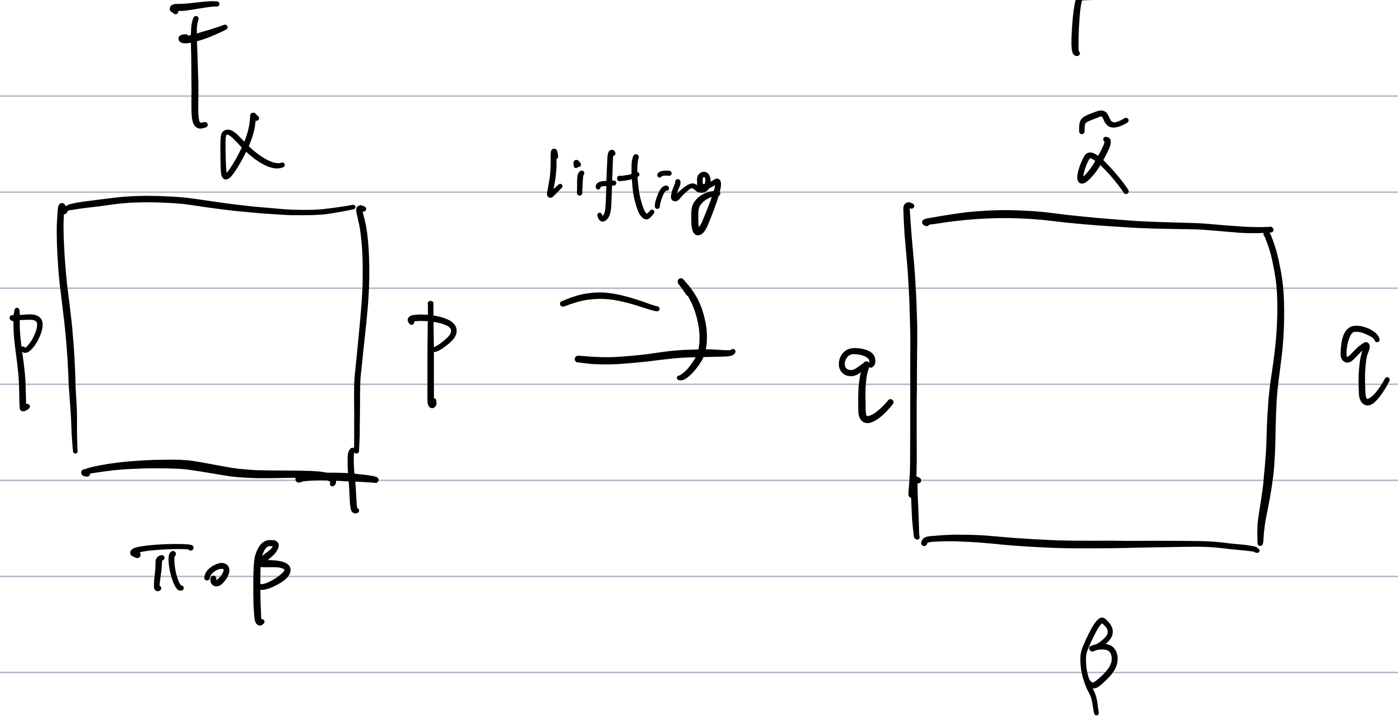
then $\tilde{\alpha}(1) = q \Leftrightarrow \langle \alpha \rangle \in \pi_*(H)$

$\Rightarrow$: ✓

$\Leftarrow$: $\exists \langle \beta \rangle \in H$

$\alpha \underset{\sim}{\overset{F}{\simeq}} \pi \circ \beta$

$\tilde{F}$



$\exists\ \tilde{\alpha}(1) = q$

We have proved that

$\pi_*(H) \leq G$ left coset.

Then $\pi_*(H) \backslash G = \{[g]\}$ $\pi_*(H) \cdot g$

$[g] = [h] \iff g \sim h$

$[g] = [h] \Leftrightarrow g \sim h$

$\Leftrightarrow \exists f \in \pi_*(H) \quad g = fh.$

Theorem. $\pi_*(H) \backslash G \overset{1:1}{\longleftrightarrow} \pi^{-1}(p)$

$\forall \langle \gamma \rangle \in G = \pi_1(p, X)$ [$\tilde{X}$ is path connected]

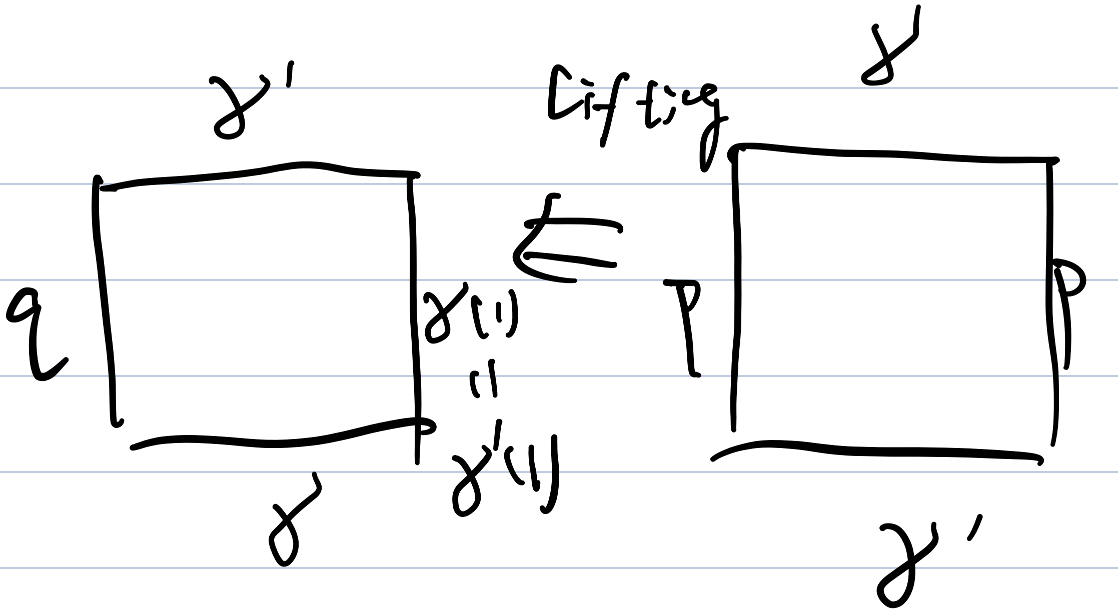
choose $\gamma \in \Omega(p, X)$

$\exists! \ \tilde{\gamma} : [0,1] \to \tilde{X}$

$\tilde{\gamma}(0) = q.$

$\phi(\langle \gamma \rangle) = \tilde{\gamma}(1) \in \pi^{-1}(p)$

well defined :.

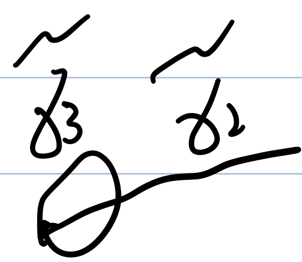


√.

If $\langle\gamma_1\rangle, \langle\gamma_2\rangle \in \pi_1(p, X)$

$$\langle\gamma_1\rangle = \langle\gamma_3\rangle\langle\gamma_2\rangle \quad \text{s.t. } \langle\gamma_3\rangle \in \pi_*(H)$$

$$\Rightarrow \tilde{\gamma}_3 \in \Omega(q, \tilde{X})$$



$$\langle \pi \circ (\tilde{\gamma}_3 \cdot \tilde{\gamma}_2)\rangle = \langle\gamma_1\rangle$$

$\Rightarrow \phi(\gamma_1) = \phi(\gamma_2)$

Theorem. $\pi : \tilde{X} \to X$, $\tilde{X}$ path connected

$\pi(q_1) = \pi(q_2) = p$

$\pi_*(\pi_1(q_1, \tilde{X})) \subseteq \pi_1(p, X)$

$\pi_*(\pi_2(q_2, \tilde{X})) \subseteq \pi_1(p, X)$

are conjugate to each other.

proof. $\gamma(0) = q_1$, $\gamma(1) = q_2$

γ_*

$$\begin{tikzcd} \pi_1(q_1, \tilde{X}) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \pi_1(q_2, \tilde{X}) \arrow[d] \\ \pi_1(p, X) \arrow[r] & \pi_1(p, X) \end{tikzcd}$$

Now we make a stronger assumption:

Y is locally path connected

i.e. $\forall y \in Y, \forall u \in N(y, Y)$, $\exists\, y \in V \subseteq u$, $V \in N(y, Y)$, s.t. V is path connected

Theorem:

If $\pi: \tilde{X} \to X$ is covering map

$\tilde{X}, X$ are path-connected

Y is path connected and

locally path-connected

$f(r) = p$

Then $f: Y \to X$ can be lifted i.e.

$\exists! \tilde{f}: Y \to \tilde{X}$, s.t. $\tilde{f}(r) = q$ and

$\pi \tilde{f} = f$

$\Leftrightarrow$ $f_*(\pi_1(y, Y)) \subseteq \pi_*(\pi_1(q, X))$

$\Rightarrow$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

$\Leftarrow$: $\forall y \in Y$, $\exists \gamma : [0,1] \to Y$

$\gamma(0) = r$, $\gamma(1) = \underline{\underline{y}}$

then $f \circ \gamma$ is a path in X.

By path-lifting thm.

$\exists \tilde{\gamma} : [0,1] \to X$

$\pi \circ \tilde{\gamma} = f \circ \gamma$

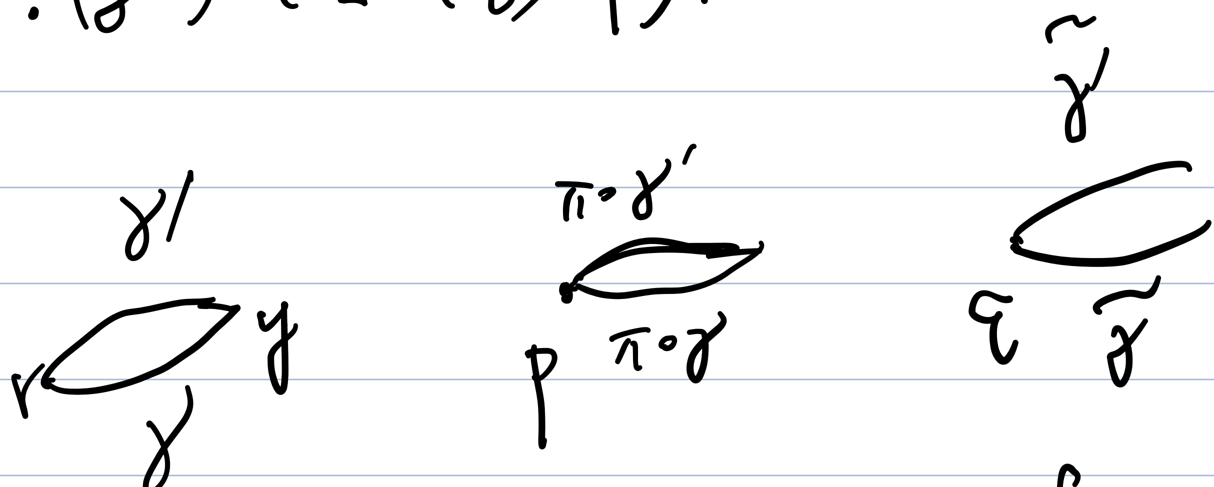
we then define $\tilde{f}(\gamma) = \tilde{\gamma}(1)$

well-defined:

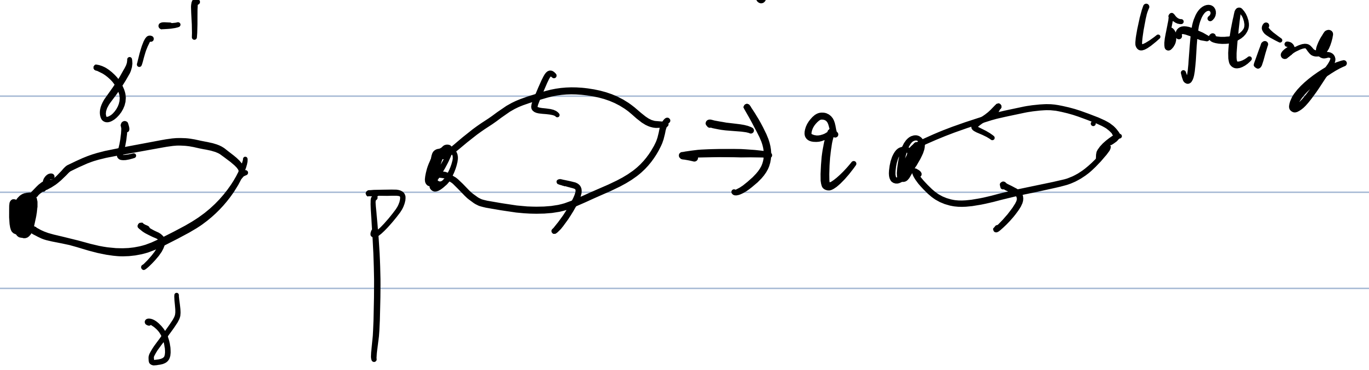
If $\gamma' : [0,1] \to Y$

$\gamma'(0) = r \quad \gamma'(1) = y$

$\gamma \cdot (\gamma')^{-1} \in \Omega(\gamma, Y)$.



uniqueness of path



Continuty :

$\square$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[ur, dashed] \arrow[r] & X \end{tikzcd} \iff \begin{tikzcd} & \pi_1(q, \tilde{X}) \arrow[d] \\ \pi_1(r, Y) \arrow[ur, dashed] \arrow[r] & \pi_1(p, X) \end{tikzcd}$$

Cor. If $\pi_1 : \tilde{X}_1 \to X$

$\pi_2 : \tilde{X}_2 \to X$

are covering map

$X, \tilde{X}_1, \tilde{X}_2$ are path-connected and

locally path-connected

$\exists\ \pi : \tilde{X}_2 \to \tilde{X}_1$ s.t. $\pi_1 \circ \pi = \pi_2$

$\Leftrightarrow \pi_2^*(\pi_1(q_2, \tilde{X}_2)) \subseteq \pi_1^*(\pi_1(q_1, \tilde{X}_1))$

$$\begin{tikzcd}
\tilde{X}_1 \arrow[dr, "\pi_1"'] & & \tilde{X}_2 \arrow[ll, "\pi"'] \arrow[dl, "\pi_2"] \\
& X &
\end{tikzcd}$$

Universal covering:

Def. we say that

$$\begin{array}{ccc} \tilde{X}_1 & \text{and} & \tilde{X}_2 \\ \downarrow & & \downarrow \\ X & & X \end{array}$$ is equivlent

$\Leftrightarrow$ $\exists\ \tilde{X}_2 \xrightarrow{\pi} \tilde{X}_1$ covering map

and $\exists\ \tilde{X}_1 \xrightarrow{\pi'} \tilde{X}_2$ covering map

and $\pi \circ \pi' = \pi' \circ \pi = Id$.

(Definition in category theory)

$\Leftrightarrow$ $\exists$ homeomorphism

$$\tilde{X}_2 \xrightarrow{\pi} \tilde{X}_1$$

We have an ambiguity of base pt

$$\begin{tikzcd}
q_2 \in \tilde{X}_2 \arrow[dr] & & \tilde{X}_1 \ni q_1 \arrow[ll, "\pi"'] \arrow[dl] \\
& \tilde{X} &
\end{tikzcd}$$

$p \in$

$\pi(q_1) = q_2$ then $\pi_{1*}(\pi_1(q_1, \tilde{X}_1))$

$= \pi_{2*}(\pi_2(q_2, \tilde{X}_2))$

$\Updownarrow$

π is an equivlence

If $\pi(q_1) \neq q_2$

equivalent $\Leftrightarrow$ $\pi_{1*}(\pi_1(q_1, \tilde{X}_1))$

$\Big\updownarrow$ conjugate

$\pi_{2*}(\pi_1(q_2, \tilde{X}_2))$

Def. If $\pi: \tilde{X} \to X$ is a covering map. $h: \tilde{X} \to \tilde{X}$ is called a covering transform

$$\begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "\pi"'] & & \tilde{X} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

$\Leftrightarrow$ • h is a homeomorphism

• The diagram above is commutes.

Theorem.

If $\pi_*(\pi_1(q, X))$ is a normal

subgroup of $\pi_1(p, X)$

Then the group of covering transform K acts on $\tilde{X}$ freely

(freely: 'stabilizer of elements is trivial).

and X is homeomorphic to $\tilde{X}/K$.

$K \curvearrowright \tilde{X} \xrightarrow{\pi} X$ (without normalness).

and $K \xrightarrow{\sim} \pi_1(p,X)/\pi_*(\pi_1(q,\tilde{X}))$

Pf:

K acts freely by the uniqueness of map lifting (without normalness). ✓.

$$\begin{tikzcd}
\tilde{X} \arrow[rr] \arrow[dr, "\pi"'] & & \tilde{X}/K \arrow[dl, "\pi'"] \\
& X &
\end{tikzcd}$$

π' is well defined because if

$x_1, x_2 \in \tilde{X}$, $\exists h \in K$, s.t. $x_1 = h(x_2)$

then $\pi(x_1) = \pi(h(x_2)) = \pi(x_2)$.

π' is continuous because π is.

(universal property of identification)

If $\pi'[q_1] = \pi'[q_2]$

$\Rightarrow \pi(q_1) = \pi(q_2)$

$p := \pi(q_1)$

$\pi_*(\pi_1(q_1, \tilde{X}))$ is normal

$\Rightarrow \pi_*(\pi_1(q_1, \tilde{X})) = \pi_*(\pi_1(q_2, \tilde{X}))$

$\Rightarrow \exists h \in K$, s.t. $q_1 = h(q_2)$

$\Rightarrow [q_1] = [q_2]$

$\Rightarrow$ injective.

Recall that all the covering map

are onto

$\forall p \in X$, $\exists q \in \tilde{X}$, $\pi(q) = p$, then

$\pi'[q] = p$.

$\forall p \in X$ $\exists u$ open, $p \in u$

$\pi^{-1}(u) = \bigcup_{\alpha} u_\alpha$

$\pi|_{u_\alpha} : u_\alpha \to u$ is homeomorphism

chose $q \in \pi^{-1}(u)$

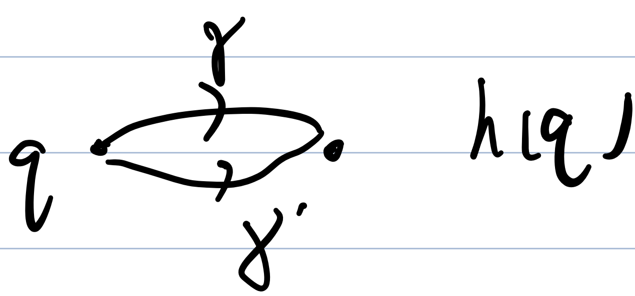
$q \in U_{\alpha_0}$

$\Rightarrow \pi_0'' (\pi|_{U_{\alpha_0}})^{-1} = (\pi')^{-1}|_U$

$\Rightarrow (\pi')^{-1}$ is continuous by glueing lemma.

$\Rightarrow X \overset{\sim}{\to} \tilde{X}/K$

Fix $p, q.$, $h \in K$



$$[\pi \circ \gamma] \in \pi_1(p, X) \big/ \pi_*(\pi_1(q, \tilde{X}))$$

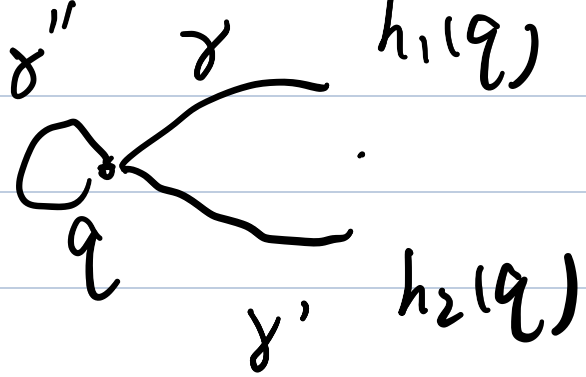
is well defined.

$$\phi : K \to \pi_1(p, X) \big/ \pi_*(\pi_1(q, \tilde{X}))$$

$$h \mapsto [\pi \circ \gamma]$$

ϕ is injective because if

$$\phi(h_1) = \phi(h_2)$$



$\exists \gamma''$, s.t.

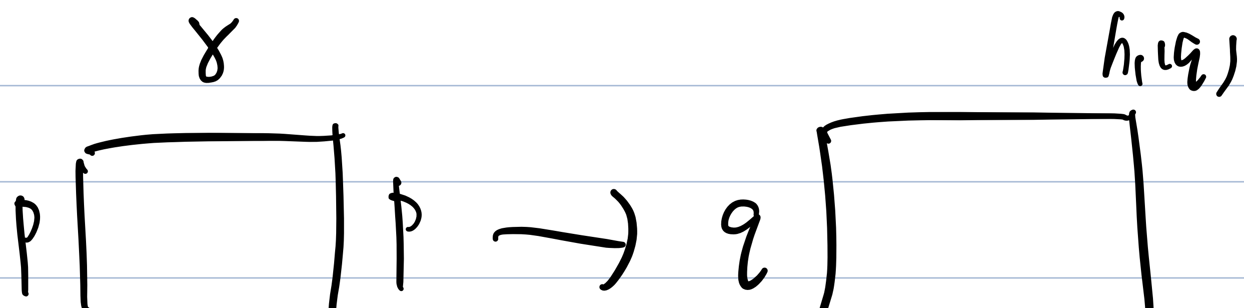
$$[\pi\circ\gamma] = [\pi\circ\gamma'']\cdot[\pi\circ\gamma']$$

$$\in \pi_1(p, X)$$

$$\Rightarrow \pi\circ\gamma \overset{F}{\simeq} \pi\circ(\gamma'\cdot\gamma'')$$

rel $\{0,1\}$.

$$\Rightarrow \gamma \overset{\tilde{F}}{\simeq} \gamma'\cdot\gamma'' \quad \text{rel } \{0,1\}.$$



$\Rightarrow \quad h_1(q) = h_2(q)$

$\Rightarrow \quad h_1 = h_2$

$\Rightarrow$ injective.

ϕ is onto:

path-lifting.

and use normal.

$[\gamma] \to \tilde{\gamma}$

$\tilde{\gamma}(0) = q$ $\qquad$ $\tilde{\gamma}(1) \in \pi^{-1}(p)$

$\exists\, h \in K, \quad h(q) = \tilde{\gamma}(1)$

If $\pi : \tilde{X} \to X$ is a covering

and $\tilde{X}$ is path-connected

Then we call $\tilde{X}$ a universal

covering space, if X is simply

connected

$$\exists! \quad \begin{tikzcd} \tilde{X} \arrow[rr, "\exists \pi''"] \arrow[dr, "\pi"'] & & \tilde{X}' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

π'' is a covering map.

$\pi' \circ \pi'' = \pi$

more over K = covering

Theorem.

If X is path-connected,

Locally path connected, semi-locally simply connected

$\left(\forall p \in X, \exists U \in \mathcal{N}(p, X) \text{ s.t.} \right.$

U is simply-connected)

then $\exists$ universal cover $\pi : \tilde{X} \to X$

Proof.

See textbook.

Recall that if G is a group

$$\langle X \rangle := \bigcap_{\substack{X \subseteq H \\ H \leq G \\ \text{subgroup}}} H$$

If $\langle X \rangle = G$, X is a set of generators of G

Free group.

$$F(X) = \{ x_1^{n_1} \cdots x_m^{n_m} \mid n_i \in \mathbb{Z},\ x_i \in X \}$$

$$(X_1^{n_1} \cdots X_m^{n_m}) \cdot (Y_1^{n_1'} \cdots Y_t^{n_t'})$$

$$= X_1^{n_1} \cdots X_m^{n_m} Y_1^{n_1'} \cdots Y_t^{n_t'}$$

If $G = \langle X \rangle$

$F(X) \to G$ surjective.

In general. $\phi : F \twoheadrightarrow G$.

$G \xrightarrow{\sim} F(X) / \ker \phi$.

$\ker \phi \trianglelefteq F(X)$

$N \subseteq F(X)$, s.t. $\ker \phi$ is the smallest normal subgroup of $F(X)$

$$\ker\phi = \bigcap_{\substack{H \trianglelefteq F(X) \\ N \subseteq H}} H$$

$$G = \langle X \mid N \rangle .$$

Example. $G = \langle \mathbb{Z}, + \rangle$

$\mathbb{Z} = \langle 1 \rangle$

$G = \mathbb{Z}/n\mathbb{Z}$

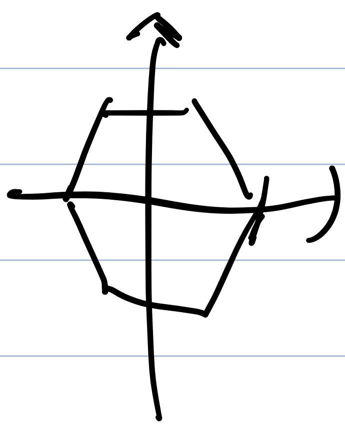
$x = [1]$

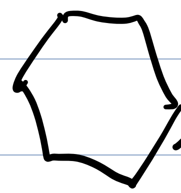
$\phi : F(X) \twoheadrightarrow G$

$\mathrm{Ker}\, F(X) = \langle x^n \rangle$

$\Rightarrow \mathbb{Z}_n = \langle X \mid x^n \rangle$

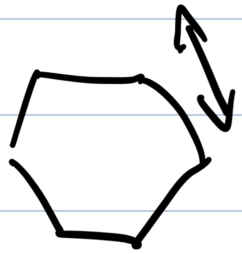
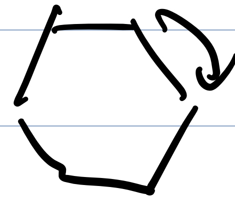
Dehedral group.



$D_{2n} = \mathrm{Aut}($  $)$

$= \{$ linear transformations

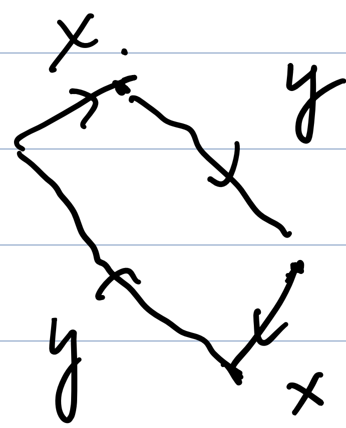
turn  into itself $\}$.

$y =$  $x =$ 

$D_{2n} = \langle x, y \rangle$

$\phi : F(\{x, y\}) \to D_{2n}.$

$y^2, x^n, xyxy \in \ker\phi$



$x^{n_1} y^{n_2} x^{n_3} \cdots y^{n_m} \in \ker\phi.$ $xy = yx^{n-1}$

$\Rightarrow x^{a} y^{b} \in \ker\phi.$

$\Rightarrow D_{2n} = \langle x, y \mid y^2, x^n, xyxy \rangle$

Example.

$G \leq \mathrm{Iso}(\mathbb{R}^n)$

$$G = \left\langle \begin{matrix} t(x,y) = (x+1, y) \\ u(x,y) = (-x+1, y+1) \end{matrix} \right\rangle$$

$G = \langle t, u \mid u^2 = tutu \rangle$

$\Rightarrow \pi_1(\text{Klein bottle}) = G$.

Free product.

Remark. $G = \langle X \mid 1 \rangle$.

and H another group.

$$\begin{tikzcd} & G \arrow[d, dashed, "\exists !"] \\ F(X) \arrow[ur] \arrow[r, "f"'] & H \end{tikzcd}$$

$\exists !\ G \dashrightarrow H$, if $N \subseteq \ker f$.

If G, H are groups, The free product $G * H$ is defined by

$\{x_1 x_2 \cdots x_m \mid x_1, \cdots, x_m \in G \sqcup H\} \cup \{e\}$.

disjoint union.

$(x_1 \cdots x_n)(y_1 \cdots y_m)$

if $x_n \in G$, $y_1 \in H$

or $x_n \in H$, $y_1 \in G$

$x_1 \cdots x_n y_1 \cdots y_m$

or we get

$x_1 \cdots x_{n-1} (x_n y_1) y_2 \cdots y_m$

Remark. We can do free product to infinite groups.

If G and H and K are groups.

$$\begin{tikzcd}
H \arrow[dr] \arrow[drr, bend left] & & \\
& H * G \arrow[r, dashed, "\exists !"] & H * G. \\
G \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

coproduct.

Lemma.

If $G = \langle X_1 \mid N_1 \rangle = F(X_1)/\langle N \rangle$

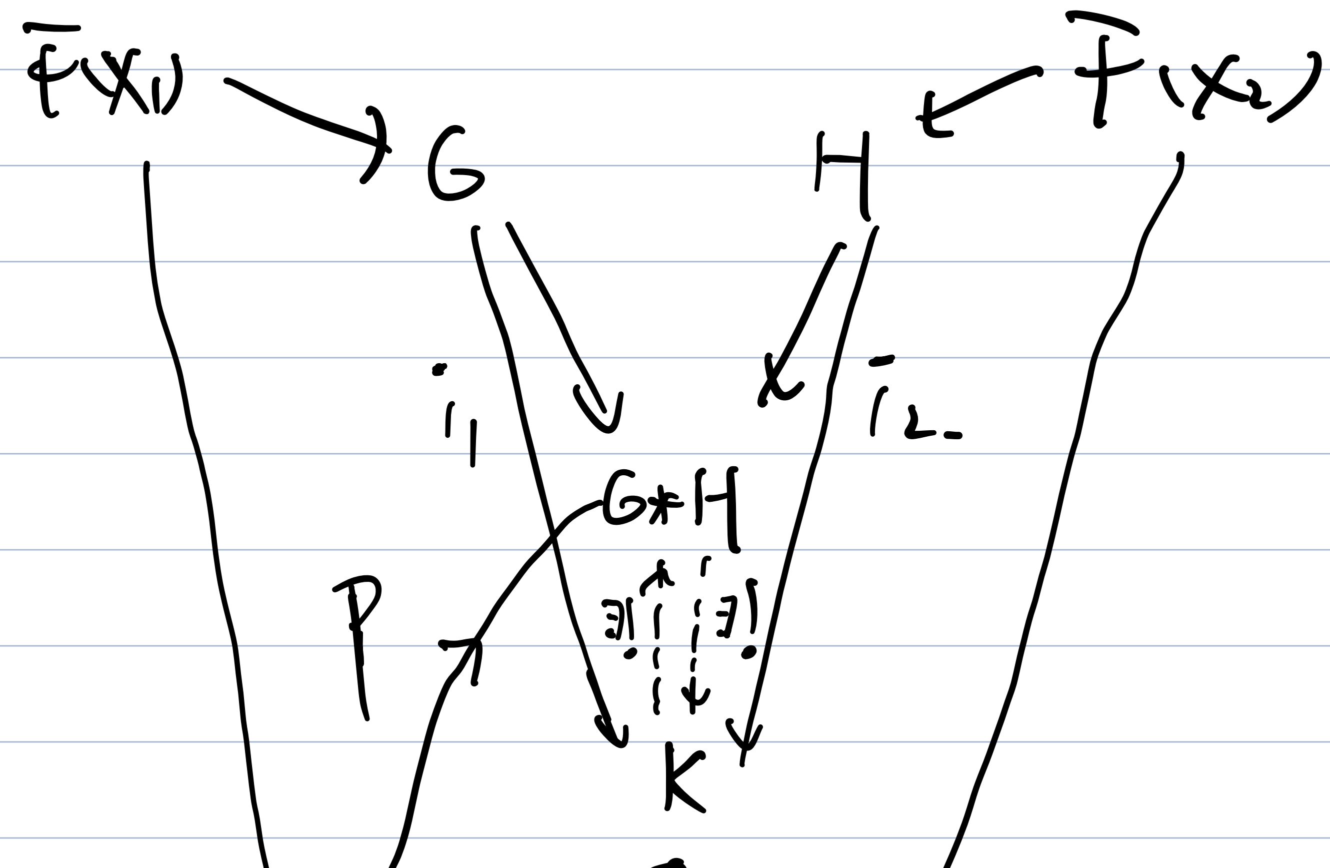
normal.

$H = \langle X_2 \mid N_2 \rangle$

Then $G * H = \langle X_1 \sqcup X_2 \mid N_1, N_2 \rangle$

$\quad = K$

Pf:



$$\begin{tikzcd} F(X_1 \sqcup X_2) \\ \langle N_1, N_2 \rangle \arrow[u, "i"] \end{tikzcd}$$

Van Kampen. Thm.

pushout.

$$\pi_1(U \cup V) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

$$:= \pi_1(U) * \pi_1(V) / \langle \pi_1(U \cap V) \rangle$$

normal subgroup.

$$
\begin{tikzcd}
 & \pi_1(U) \arrow[dr] \arrow[ddl] & \\
\pi_1(U\cap V) \arrow[ur] \arrow[dr] & & \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \arrow[ddll, dashed, "k"] \\
 & \pi_1(V) \arrow[ur] \arrow[dl] & \\
G & &
\end{tikzcd}
$$

$$\mathbb{Z}\times\mathbb{Z} = \langle g, h \mid ghg^{-1}h^{-1} \rangle$$

In general, if G is a group,

the Abelianization of G is

$$\mathrm{Abel}(G) := G/N$$

$N = \langle ghg^{-1}h^{-1} \rangle$ normal subgroup.

Prop. $\mathrm{Abel}(G \times H) = \mathrm{Abel}(G) \times \mathrm{Abel}(H)$

Prop. G is a group

H is a Abelian group

$$\forall \begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \\ \mathrm{Abel}(G) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Example: $H_1(X, \mathbb{Z}) = Abel(\pi_1(P, X))$

$$Hom(H_1(X, \mathbb{Z}), G) = H^1(X, G)$$

Theorem. Van Kampen theorem.

If X is a topological space,

U, V are open, $X = U \cup V$

If $U, V, U \cap V$ are path-connected

$\Rightarrow \pi_1(P, X) =$

$(P, U) * \pi_1(P, V) / \langle (i_U)_*(x) = (i_V)_*(x) \rangle$

$\pi_1(p,U) * \pi_1(p,V) / \langle (i_U)_* \ \cdots \ (i_V)_* \ \cdots \rangle$

$$\begin{tikzcd} & \pi_1(U) \arrow[dr, "i_1"] & \\ \pi_1(U\cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(U\cup V). \\ & \pi_1(V) \arrow[ur, "i_2"'] & \end{tikzcd}$$

Proof: $(i_1)_* : \pi_1(p,U) \to \pi_1(p,X)$

$(i_2)_* : \pi_1(p,V) \to \pi_1(p,X)$

$\Rightarrow \exists\, \Phi : \pi_1(p,U) * \pi_1(p,V) \to \pi_1(p,X)$

$\forall \langle \gamma \rangle \in \pi_1(p,U\cap V)$

$(i_1)_*(i_U)_* \langle \gamma \rangle = (i_2)_*(i_V)_* \langle \gamma \rangle$

$$\phi: \pi_1(p,U) * \pi_1(p,V) \big/ \langle (i_U)_* \langle\gamma\rangle = (i_V)_* \langle\gamma\rangle \rangle \longrightarrow \pi_1(p,X)$$

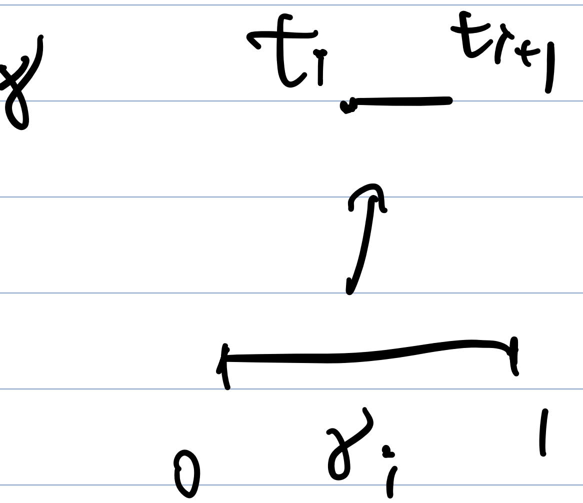
$\forall\ \gamma \in \Omega(p,X)$

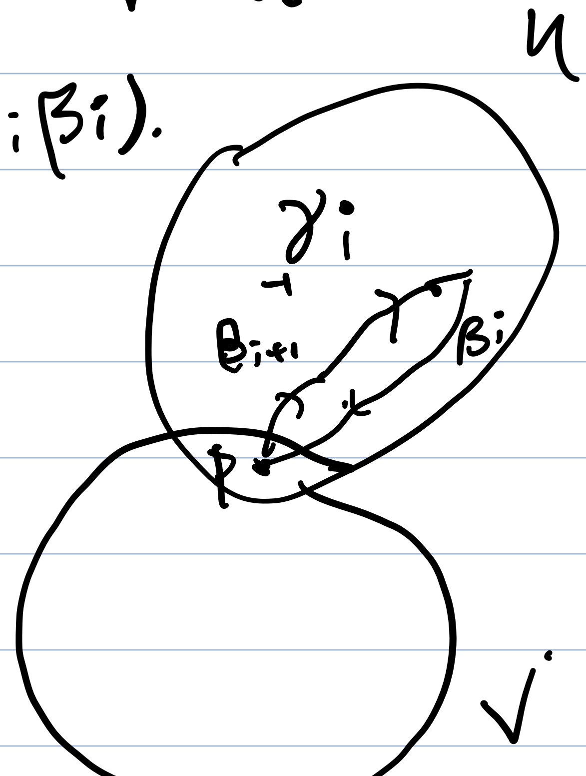
Lebesgue's Lemma

$\exists\ 0 = t_0 < t_1 < \cdots < t_n = 1$

$\gamma([t_i, t_{i+1}]) \subseteq V$ or U

$(\beta_{i+1}^{-1}\gamma_i \beta_i)$.





$\Rightarrow \phi$ is surjective.

If $\langle \gamma_1 \cdots \gamma_n \rangle = \langle e \rangle$

$\gamma_i \in \pi_1(P, U)$ or $\pi_1(P, V)$

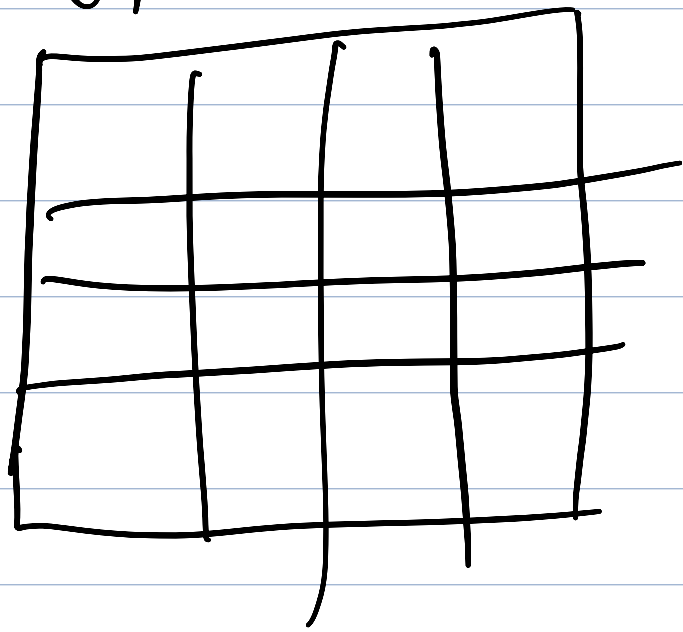
$\gamma_1 \sim \gamma_n \sim e \quad \text{rel } \{0,1\}$

$F : [0,1] \times [0,1] \longrightarrow X$

$F^{-1}(U)$, $F^{-1}(V)$ are open.

Lebesgue number: $\varepsilon > 0$.

γ_1 γ_2



shrinking γ_i, s.t.

$\gamma_i \subseteq V \setminus U$ or $U \setminus V$ or $U \cap V$.

……

See hatcher.

$\Box$

Van Kampen's theorem.

S^n.

$u, V \xrightarrow{\sim} \mathbb{R}^n$

$u \cap V \xrightarrow{\sim} S^{n-1} \times (-\varepsilon, \varepsilon)$

$e \qquad e.$

$\Rightarrow \pi_1(p, S^n) = \pi_1(p, u) * \cancel{\pi_1(p, V)}$

$\square$

$\Rightarrow \pi_1(p, S^n) = \{e\}, \ n \geq 2.$

If $J \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$

$J = S^1$

$\Rightarrow \quad \pi_1(p, \mathbb{R}^2 \setminus J) = \mathbb{Z}$

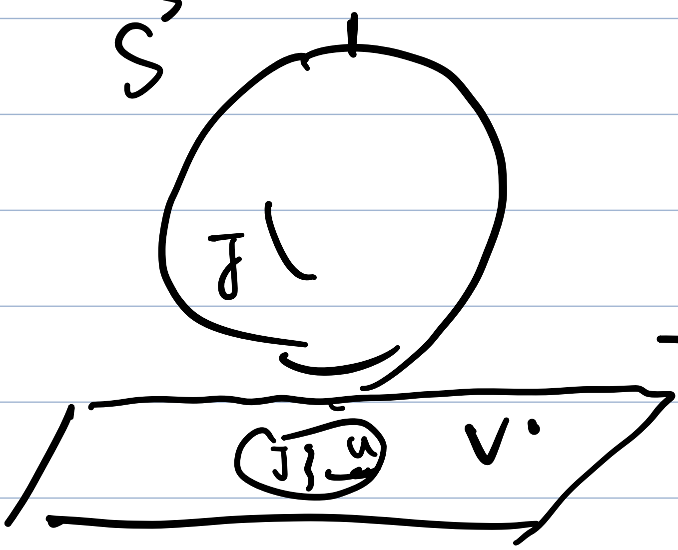
Proof: $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

$U = B(0, 2R) \setminus J \qquad V = \underline{\{r > R\} \cup \{\infty\}}$.

$\quad\quad\quad\quad\quad\quad V$

$U \xrightarrow{\sim} \mathbb{R}^3 \setminus J$.

$S^3 \qquad\qquad U \cap V = S^2 \times (-\varepsilon, \varepsilon)$



$\pi_1(p, U \cap V)$

$= \pi_1(p, S^2) \times \pi_1(p, (-\varepsilon, \varepsilon))$

$= \langle e \rangle$

van kam pen

$$\Rightarrow \pi_1(S^3 \setminus J) = \pi_1(\mathbb{R}^3 \setminus J)$$

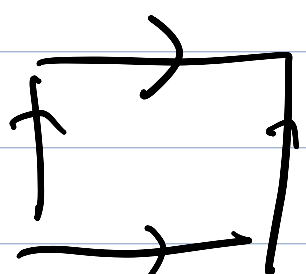
$$= \pi_1(\mathbb{R}^3 \setminus \mathbb{R})$$

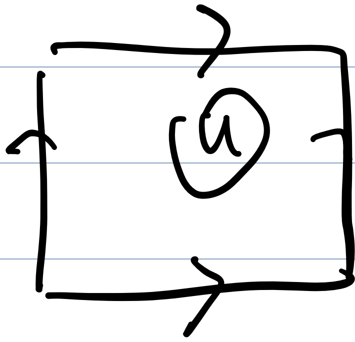
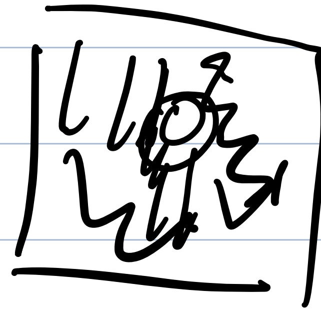
$$= \pi_1(\mathbb{R}^2 \setminus \{0\} \times \mathbb{R})$$

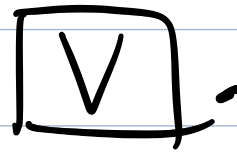
$$= \pi_1(S^1 \times (0, +\infty) \times \mathbb{R})$$

$$= \mathbb{Z}$$

Example.

$\pi_1\Big($  $\Big)$

$U =$  $V =$ 

$U \cap V = S^1 \times (-\varepsilon, \varepsilon)$ 

Definition.

If X, Y are Top.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

f, g are continuous.

If $f \circ g \overset{F}{\simeq} Id$

$g \circ f \overset{G}{\simeq} Id$

Then X is homotopic to f.

Example.

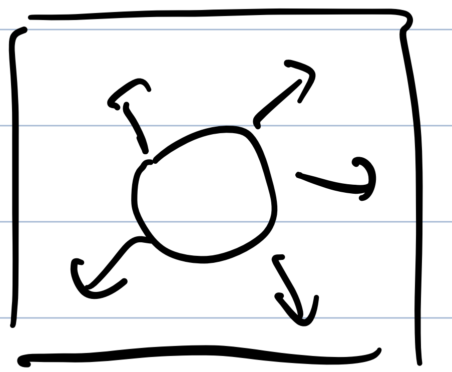
$A \subseteq X$ $\quad G : X \times [0,1] \to X$

$G(x,0) = x$, $G(x,1) \in A$

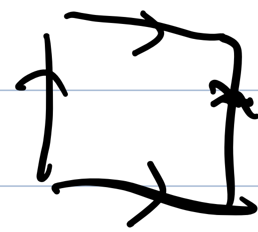
and $G(x,t) = x$ for $\forall x \in A$

then we call G a deformation retract

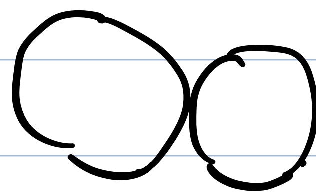
Back to the torus.



$X = V$

$A =$ 

$X \longrightarrow A$

$=$ 

If X deformation to A, then X is homotopic to A.

$$\begin{tikzcd} A \arrow[r, bend left, "i"] & X \arrow[l, bend left, "G(x,1)"] \end{tikzcd}$$

$\mathbb{Z} * \mathbb{Z}$.

A homotopic to X.

Homotopic is an equivalent relation.

If $X \sim Y$,

$\mathbb{R}^2 \setminus \{0\} \to S^1$

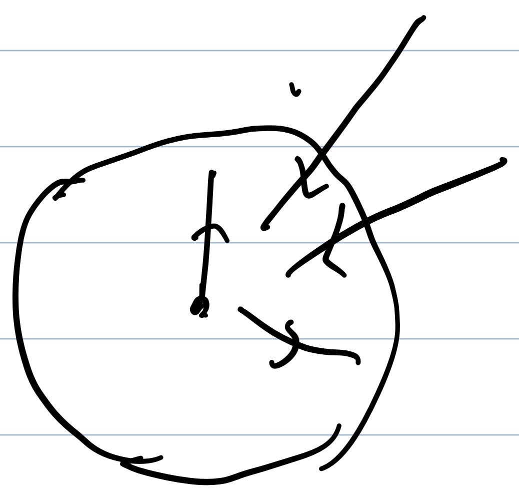
$V: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$

deformation

$\mathbb{R}^2 \setminus \{0\} \longrightarrow S^1$

retract

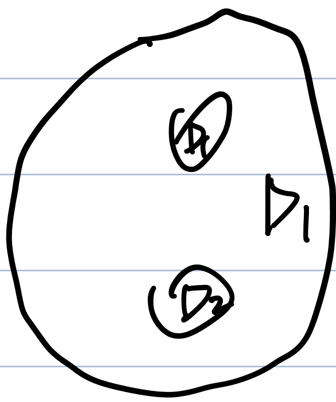
$$F(x,t) = \left(t + (1-t)|x|\right)\frac{x}{|x|}$$



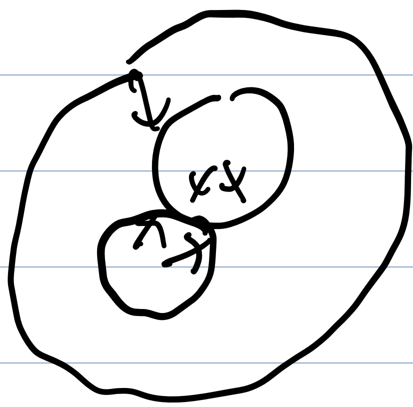
$$\frac{d(F(x,t))}{dt} = V(F(x,t))$$

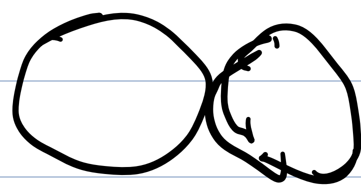
vector field.

$$V(y,t)=\frac{1-|y|}{1-t}\cdot\frac{y}{|y|}$$



$D\setminus(D_1\cup D_2)$

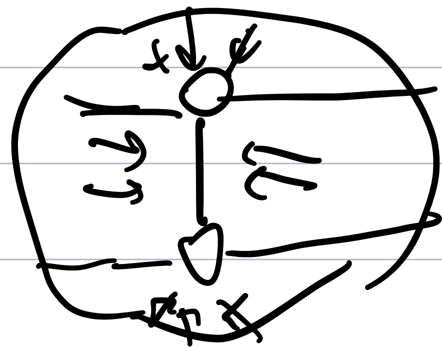


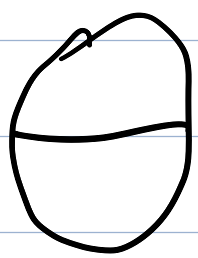


Construction:

$$F(x,0)=x \qquad F(x,1)=f(x)$$

$$F(x,t)=tf(x)+(1-t)x$$





Theorem. $f, g : X \to Y$

$$f \overset{F}{\simeq} g$$

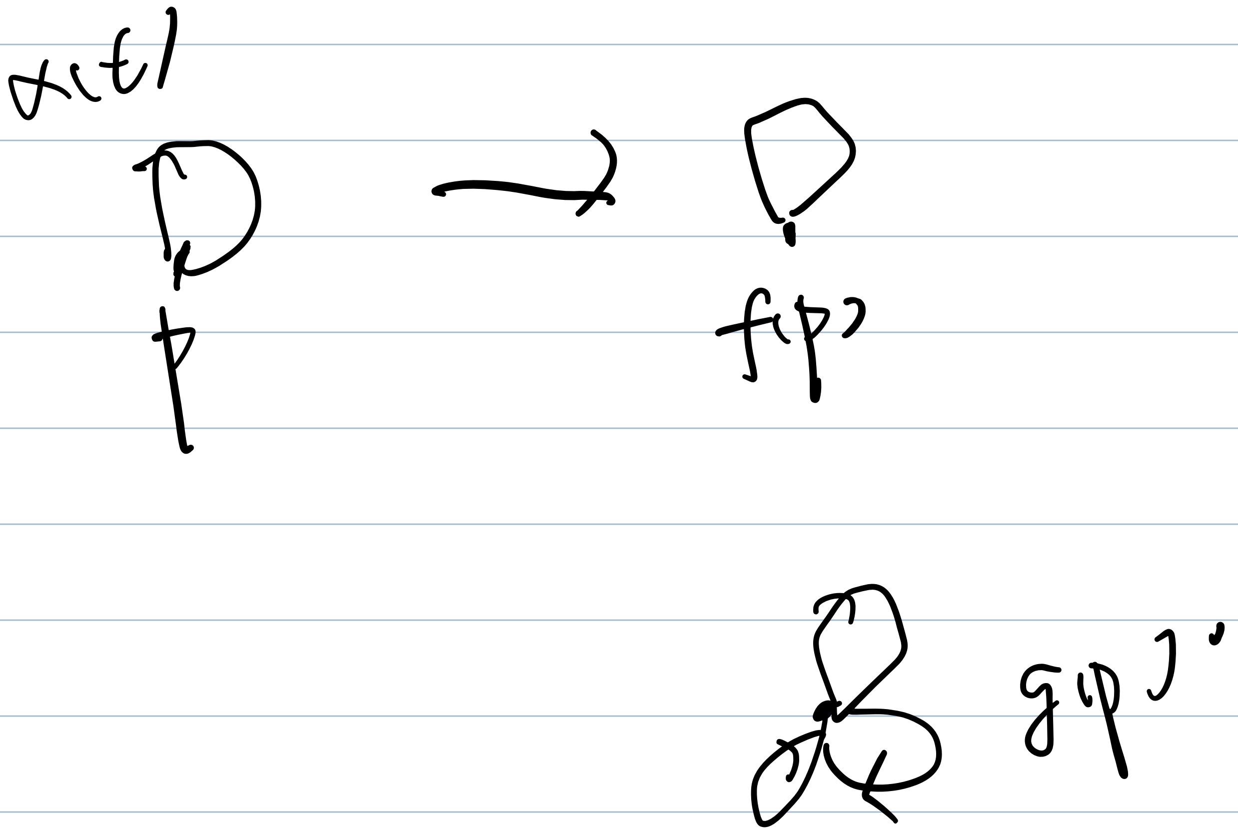
$$\pi_1(p, X) \xrightarrow{f_*} \pi_1(f(p), Y)$$

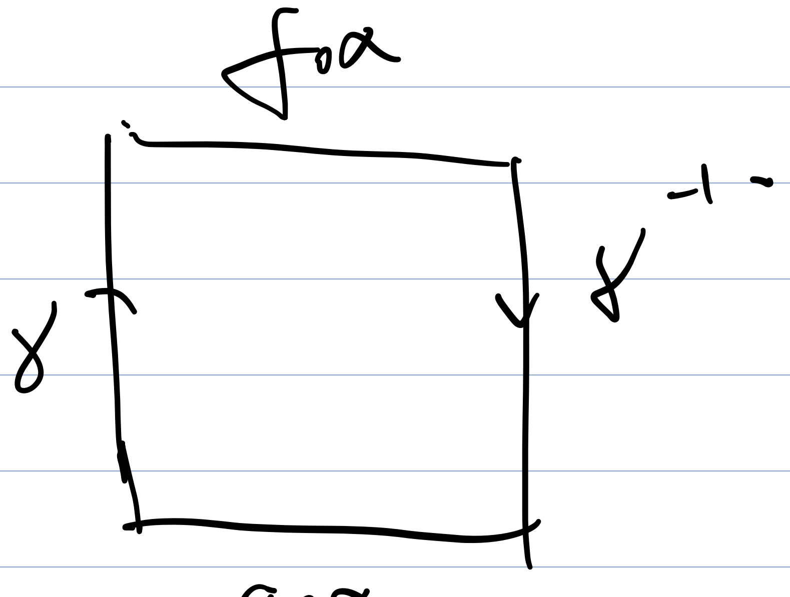
$$\Big\downarrow g_*$$

g_γ

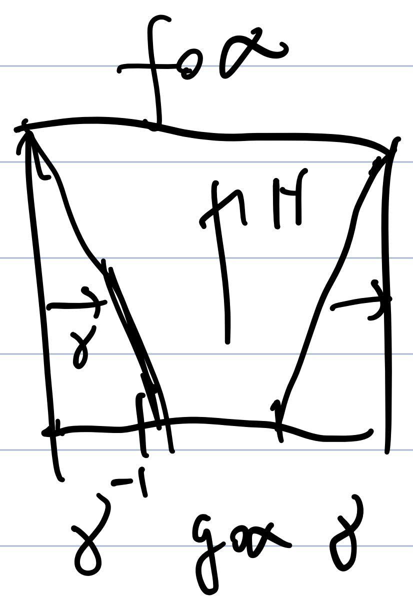
$\pi_1(g(p), \gamma)$

Where γ from $f(p)$ to $g(p)$.





$g\circ\alpha$



$f\circ\alpha \sim g\circ\alpha$.

Corollary. If $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, X, Y path connected.

are homotopic, then

$f_p : \pi_1(p, X) \to \pi_1(f(p), Y)$

is an isomorphism.

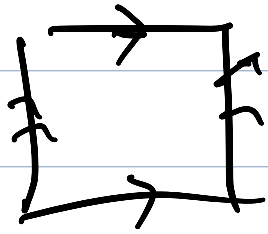
$$\begin{tikzcd}
\pi_1(p,X) \arrow[r, "f_*"] \arrow[dr, "Id"'] & \pi_1(f(p),Y) \arrow[r, "g_*"] & \pi_1(g\circ f(p),X) \arrow[dl, leftrightarrow, "\gamma^*"] \\
& \pi_1(p,X) &
\end{tikzcd}$$

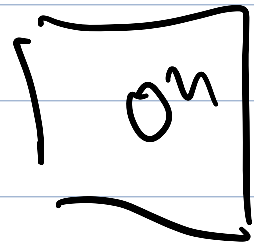
$\Rightarrow$ $g_* f_*$, $f_* g_*$ are isomorphism

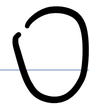
$\Rightarrow$ f_* is an bijective .

$\square$

Applications :

T^2 = 

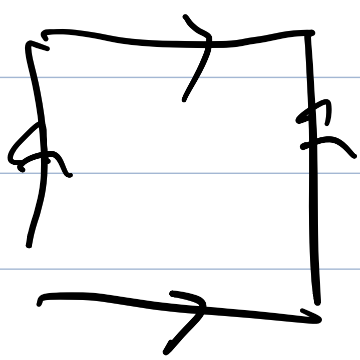
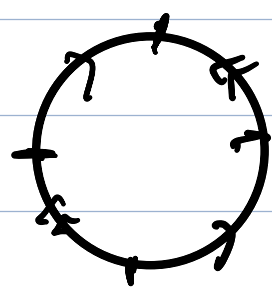
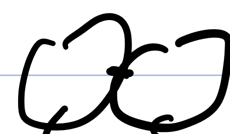
U = 

V = 

$U \cap V$ =  $\mathbb{Z}$

$\pi_1(U \cap V) = \mathbb{Z}$

U can deformation retracts

to  $\simeq$  = 

α β

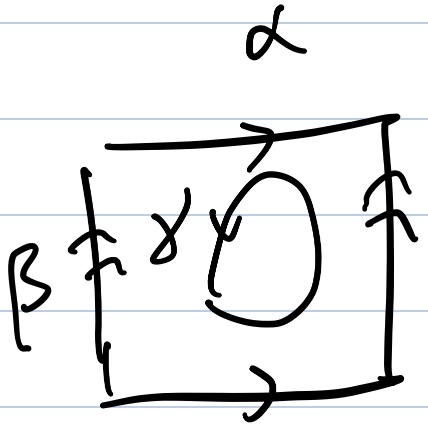
one pt

union.

$$\Rightarrow \pi_1(U) = \mathbb{Z} * \mathbb{Z}$$

$$= \langle \alpha, \beta \rangle$$

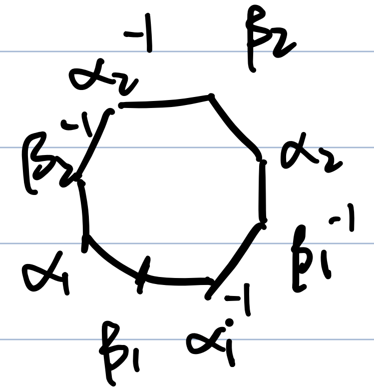
$$\Rightarrow T^2 = \langle \alpha, \beta \rangle / \langle i(\gamma) \rangle$$



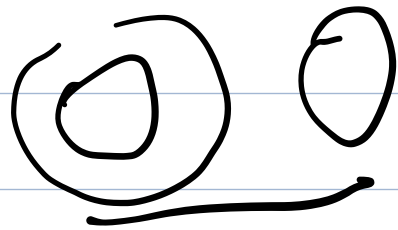
$$i(\gamma) = \alpha\beta\alpha^{-1}\beta^{-1}$$

$$\Rightarrow \pi_1(T^2) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} \rangle$$

$$= \mathbb{Z}^2$$

$\pi_1 \Big($ 

$\Big) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \,|$



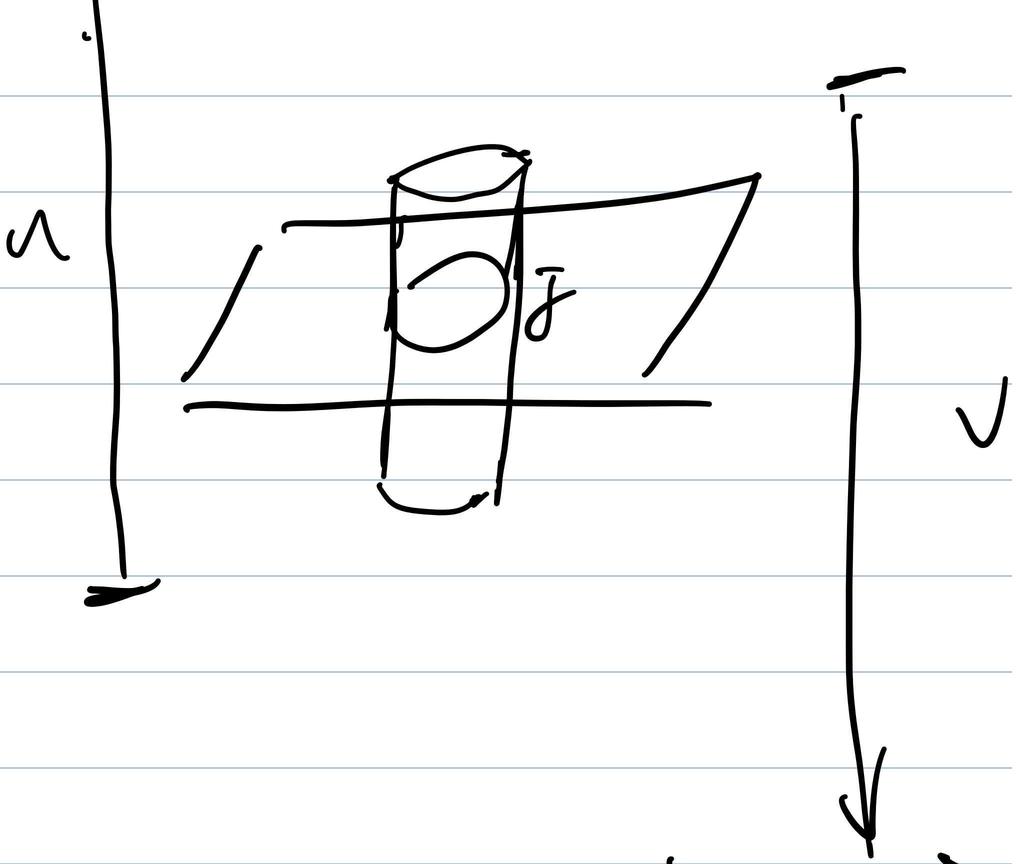
$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1}$

Weak Jordan curve theorem.

If $J \subseteq \mathbb{R}^2$, J is homeomorphic to S^1, then $\mathbb{R}^2 \setminus J$ is not connected

pf: $\mathbb{R}^2 \subseteq \mathbb{R}^3$

$\mathbb{R}^3 \setminus J =$



$$U = \{(x,y,z) \mid z > 0\}$$

$$\cup \{(x,y,z) \mid (x,y) \notin J,\ 0 \geq z > -1\}$$

$$V = \{(x,y,z) \mid z < 0\}$$

$$\cup \{(x,y,z) \mid (x,y) \notin J,\ 1 > z \geq 0\}$$

$$F(x,y,z,t) = (1-t)(x,y,z) + t\, f(x,y,z)$$

$\Rightarrow$ U, V deformation retracts to $\mathbb{R}^2$

$\Rightarrow$ $\pi_1(U) = \pi_1(V) = \langle e \rangle$

If $\mathbb{R}^2 \setminus J$ is path-connected

$U \cap V = R \setminus J \times (-1, 1)$ is

path-connected

van kampen $\Rightarrow$ $\pi_1(\mathbb{R}^3 \setminus J) = \langle e \rangle$

$S^3 = \mathbb{R}^3 \cup \{\infty\}$

$\cup\!|$

$S^2 = \mathbb{R}^2 \cup \{\infty\} \supseteq J$

$U_1 = \{\infty\} \cup \{r > 2R\}$

$U_2 = \{r < R\} \cap \mathbb{R}^3 \setminus J.$

$\Rightarrow U' \cap V' = S^2 \times (R, 2R)$

$U' \cap V' = S^3 \setminus J$

$\Rightarrow S^3 \setminus J$ is path connected.

$J \subset S^3$

Choose $p \in J$

$S^3 \setminus \{p\} = \mathbb{R}^3$

$J \setminus \{p\} \subseteq \mathbb{R}^3$

$f: S^3 \setminus \{p\} \to \mathbb{R}^3$

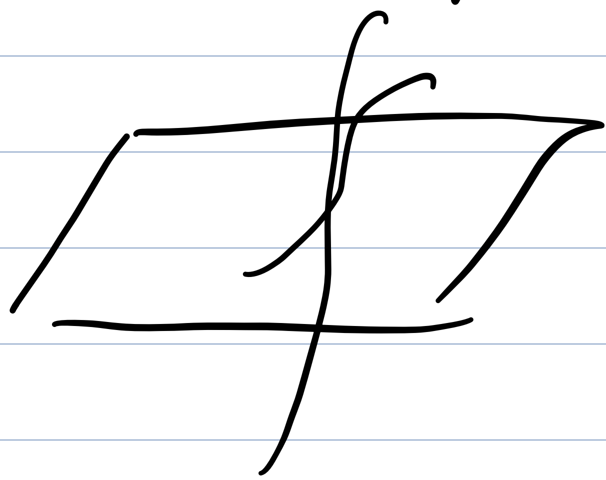
$f|_{J \setminus \{p\}} \to L$

$\Rightarrow S^3 \setminus J = \mathbb{R}^3 / L$, $L \subseteq \mathbb{R}^2$

L is homeomorphic to $S^1 \setminus \{p\} = \mathbb{R}$.

Tietze $\Rightarrow$ $g: \mathbb{R}^2 \to \mathbb{R}$

$$\mathrm{Graph}(L) := \{(x, y, g(x,y)) \mid (x,y) \in L\}$$



$$F(x,y,z) = (x, y, z + g(x,y))$$

$$\Rightarrow \quad \mathbb{R}^3 \setminus L \xrightarrow{\sim} \mathbb{R}^3 \setminus \mathrm{Graph}(g|_L)$$

$$\cong$$

$$\mathbb{R}^3 \setminus \mathbb{R}^1$$

$$=$$

$$\mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})$$

$\mathbb{R}^2 \setminus \{0\}$ deformation retracts to S^1

$\Rightarrow \pi^1(\mathbb{R}^3 \setminus L) = \mathbb{Z}$

Contradiction!

Definition.

X is contractible if

Id_X is homotopic to constant

map

Theorem.

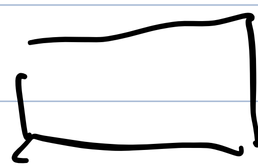
(1) Contractible $\Leftrightarrow X \sim \{p\}$

(2) Contractible $\Rightarrow$ X is simply connected.

(3) X is contractible

$f, g : Y \to X$

$\Rightarrow$ f is homotopic to g



Theorem.

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

then X is path-connected

$\Leftrightarrow Y$ is path-connected.

Pf: Suppose X is path-conneted.

$g(p), g(q) \in X$

$\exists \gamma, \ \gamma(0) = g(p), \ \gamma(1) = g(q)$

p, q

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$\Rightarrow f \circ \gamma : [0,1] \to Y$

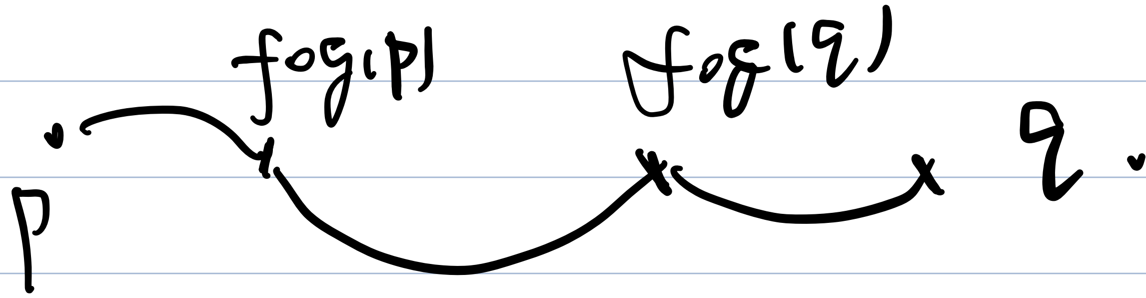
$f \circ \gamma(0) = f \circ g(p)$

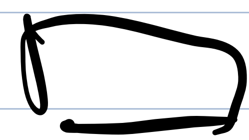
$f \circ \gamma(1) = f \circ g(q)$

$F: Y \times [0,1] \to Y$

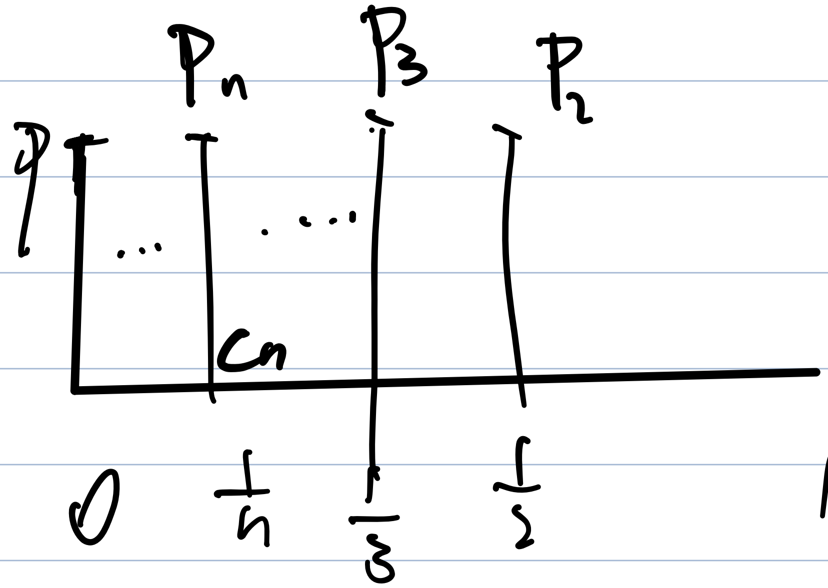
$F(x,0) = f \circ g(x)$

$F(x,1) = x$





Remark. A contractible space may not deformation retracts to $\{p\}$.



$Id_X \sim C_0 \sim C_p.$

But X can't deformation retract

to $\{p\}$

that is, $\nexists\ F: [0,1] \times X \to X$

s.t.
$$\begin{cases} F(x,0) = x, \quad F(x,1) = p \\ F(p,t) = p \end{cases}$$

Since $\{p_n\} \to p$, X is compact.

F is uniformly continuous

$\Rightarrow \forall \varepsilon > 0$, $|F(p_n, t) - \bar{F}(p, t)| < \varepsilon$

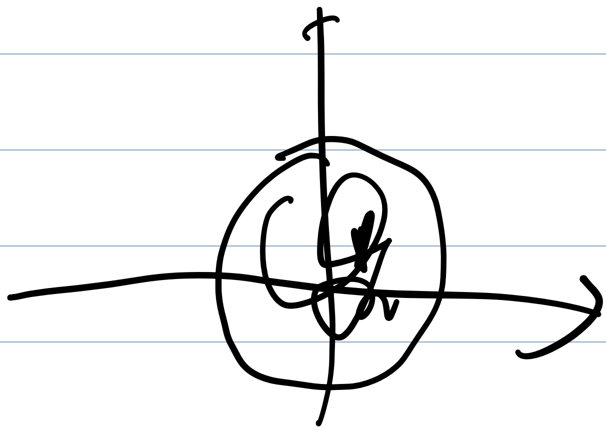
for n large enough

Brouwer fixed-pt thm.

$\forall n$. $\overline{B(0,1)} \subseteq \mathbb{R}^{n}$

If $f: \overline{B(0,1)} \to \overline{B(0,1)}$ is continuous

$\Rightarrow \exists x, f(x) = y$



Pf: $n = 1$

$\overline{B(0,1)} = [-1, 1]$

If f has no fixed pt.

$\Rightarrow [-1,1] = \{x \mid f(x) > x\} \cup \{x \mid f(x) < x\}$

Contradicts to the connectedness

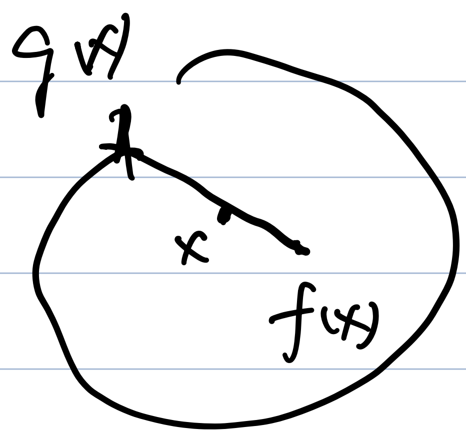
$\square$.

If $A \subseteq X$ $g: X \to A$,

$g|_A = Id_A$

Then call g a retraction.

$n=2$ If f has no fixed pt.



g(x) is defined to be the

intersect of $\overrightarrow{f(x)\,x}$ and S^1

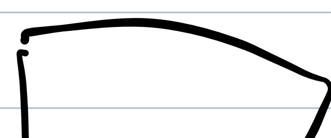
$$\overline{B(0,1)} \xrightarrow{g} S^1 \xrightarrow{i} \overline{B(0,1)}$$

g is a retraction

$\overline{B(0,1)}$ is contractible

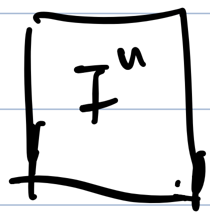
$\Rightarrow\ i g \sim Id.$

$\Rightarrow\ \pi_1(\overline{B(0,1)}) = \pi_1(S^1)$



dim n :

$\pi_n(p, X) = \{ f : S^n \to X ,\ f|_{S^{n-1}} = p \}$



$S^n = I^n / \partial I^n.$

Homotopy group.

$H_n(X)$ Homology group.

If $A \subseteq \mathbb{R}^2$

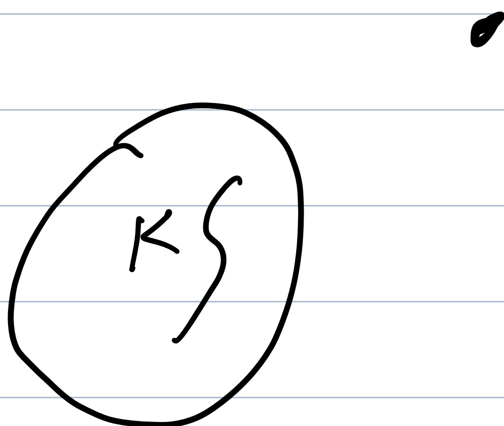
A is homeomorphic to $[0,1]$

$\Rightarrow \mathbb{R}^2 \setminus A$ is path-connected.

$A \subseteq B(0,R)$

$\Rightarrow$ There is exactly one unbounded path-connected component.

If K is a bounded path-connected component of $\mathbb{R}^2 \setminus A$



$K \subseteq B(0,R)$

Choose $p \in K$, let r be the

projection from

$$B(0,R) - \{p\} \to \partial B(0,R)$$

$$r\big|_{S^1} = Id$$

$r\big|_A$ is a continuous map

from A to S^1

Tietze extension

$\mathbb{R} \to S^1$ covering map

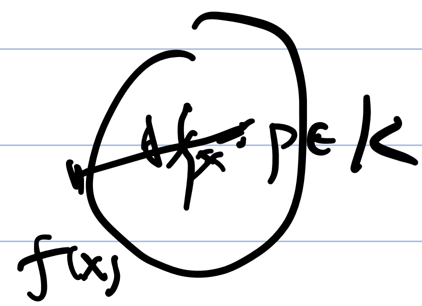
path-lifting thm

$\Rightarrow \exists f: A \to \mathbb{R}$

Tietze extension thm

$\Rightarrow$ extended to

$g: \mathbb{R}^2 \to \mathbb{R}$, $g|_A = f$



Then $h(x) = \begin{cases} \pi \circ g, & x \in K \quad \text{open} \\ \pi \circ g = r, & x \in A \quad \text{closed} \end{cases}$

$r, \quad x \in (\overline{B(0,R)} \setminus A) \setminus K$

open

A is compact. $\underline{\mathbb{R}^2 \setminus A \text{ is open}}$

its path-connected component and connected component are equivalent and are open

$\overline{K} \subseteq A \cup K$ (Since $K \subseteq \mathbb{R}^2 \setminus A$ is closed).

Similarly

$$\overline{(\overline{B(0,R)} \setminus A) \setminus K} \subseteq A \cup (\overline{B(0,R)} \setminus A) \setminus K$$

Glueing Lemma

$\Rightarrow h|_{S^1} = Id$ $\qquad S^1 = \partial \overline{B(0,R)}$

h is a retraction from $\overline{B(0,R)}$ to S^1

$\overline{B(0,R)} \xrightarrow{h} S^1 \xrightarrow{i} \overline{B(0,R)}$ ✓

Contradiction !

Manifold. $p \in U$

surface

(1) : $\exists\ \phi : U \xrightarrow{\sim} B(0,r) \subseteq \mathbb{R}^n \quad \phi(p) = 0$

(2) $\exists\, \varphi : U \xrightarrow{\sim} B(0,r) \subseteq \mathbb{R}^n_+ = \{(x_1,\dots,x_n) \mid x_n \geq 0\}$

$\varphi(p) = 0$

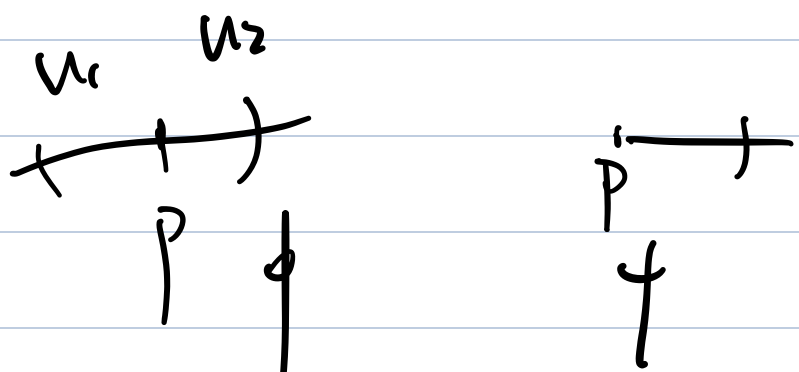
surface with boundary

How can (1), (2) both hold?: No!

dim 1:

$B(0,\varepsilon) \setminus \{0\}$ is not connected.

$= U_1 \cup U_2$

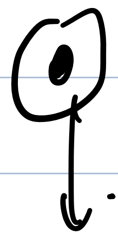
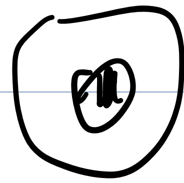


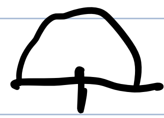
U_1, U_2 are connected $\quad U_1 \cap U_2 = \emptyset$

$\psi(\varphi^{-1}(U_1 \cup U_2)) \subseteq [0, \varepsilon)$

$u_1 \cap u_2 = \phi$ $\qquad$ X-

dim 2: $\qquad p \in u_i$

$\phi: u_1 \longrightarrow$  $\qquad$  $\sim S^1$

$\psi: u_2 \longrightarrow$ 

$B(0,\varepsilon_1) \subseteq \phi(u_1 \cap u_2)$

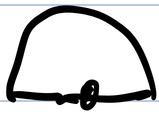
$\psi \circ \phi^{-1}(B(0,\varepsilon_1)) \supseteq B(0,\varepsilon_2)$

$r: \mathbb{R}^2 \setminus \{0\} \to S^1$

$r|: \phi \circ \psi^{-1}(B(0,\varepsilon_2)) \to S^1$

$|_Y \qquad \qquad \overset{\|}{r}$

$$(r|_Y)^*: \quad \pi_1(q, X) \to \pi_1(r(q), S^1)$$

$X =$  $\quad \boxed{\pi_1(X) = \{e\}}$

 S^1

But $\exists\, i: S^1 \to X$

$\Rightarrow \boxed{(r|_Y)^* \text{ is surjective}}$

Contradiction!

If (1) happens, call p interior pt

(2) $\qquad p$ boundary pt.

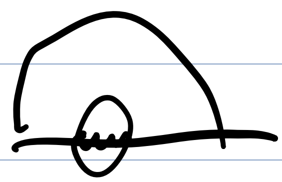
If X has no boundary, compact

call it closed manifold

(closed surface)

The boundary pts ∂X is a

closed manifold, $n-1$



X is a manifold with boundary

Then a differential structure

means

$X = \bigcup_{i \in I} U_i \quad U_i$ open

$\phi_i : U_i \to V_i \subseteq \mathbb{R}^n$ (or $\mathbb{R}^n_+$)

s.t. $\phi_i \circ \phi_j^{-1} \left(\phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \right)$

is smooth.

If $X = \bigcup_{i \in I} U_i \quad \phi_i$

$= \bigcup_{j \in J} U_j \quad \phi_j$

$\Rightarrow X = (\bigcup_i U_i) \cup (\bigcup_j U_j)$

if this is a differential

structure, we call

(U_i, ϕ_i) (U_j, ϕ_j) defines

a same differential structure.

$X = \bigcup_{i \in I} U_i$ ϕ_i

$Y = \bigcup_{j \in J} U_j$ ψ_j

$f: X \to Y$ is smooth, if

$\psi_j \circ f \circ \phi^{-1} : \phi_i(f^{-1}(U_j) \cap U_i) \to \mathbb{R}^m$

is C^∞

$X_1 = \bigcup_i u_i$ $\quad$ $X_2 = \bigcup_j v_j$

is equivalent, if Id is smooth. from $X_1 \to X_2$, $X_2 \to X_1$

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

diffemorphic 微分同胚

Question:

X is a topological manifold,

Is X has a differential structure

dim 1, 2, 3 ✓

dim 4:

✗

Counter example by Donaldson.

Question:

homeomorphism $\xrightarrow{?}$ diffemorphism

✗.

But dimension ≤ 3, this is

true

Poincare's Conjecture :

If X is homotopic to S^n, is

X diffemorphic to S^n ?

$$\Updownarrow$$

(1) $X \sim S^n \overset{?}{\not\Longrightarrow} X \xrightarrow{\sim} S^n$

(2) X is homeomorphic to S^n

$$\Downarrow ?$$

X is differmorphic to S^n

(1): True.

(2): $n = 1, 2, 3, 5, 6$ ✓

$n = 4$: ?

$n \geq 7$: ✗

Goal: use differential structure and Morse theory to classify surface.

and use fundamental group

to prove

$$X^2 \overset{\text{homotopic}}{\sim} Y^2$$

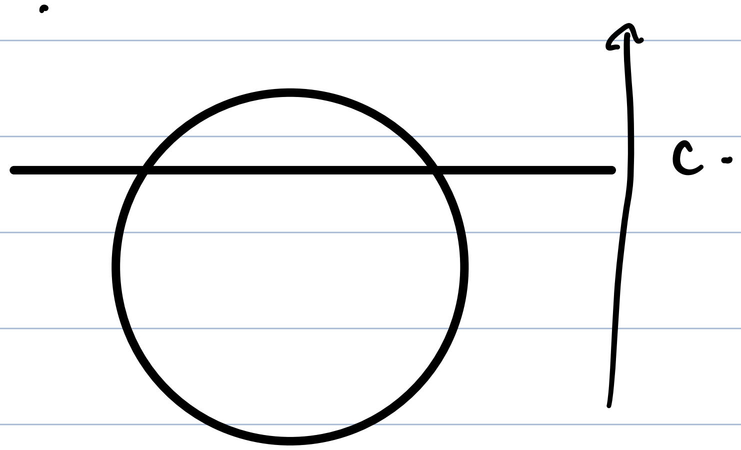
$$\Downarrow$$

$$X \xrightarrow{\sim} Y^2$$

---

dim 1 :

S^1 :



Idea : f is a smooth function on

X,

$\phi_i : U_i \to V_i$

$f \circ \phi_i^{-1}$ is smooth

If $\frac{\partial (f \circ \phi_i^{-1})}{\partial x_q} (\phi_i(p)) = 0$, $\forall q$

Call p a critical pt.

Def. f is called a Morse function, if $\forall$ critical pt,

$\mathrm{rank} \left(\frac{\partial^2 (f \circ \phi_i^{-1})}{} (\phi_i(p)) \right) = n$

$\partial x_i \partial x_j$

Theorem.

If f is a Morse function on X, then near each critical pt p,

$$\exists\, \psi_p : U_p \to V_p \subseteq \mathbb{R}^n, \text{ s.t.}$$

ψ_p is a homeomorphism,

$$\psi_p \circ \phi_i^{-1},\ \phi_j \circ \psi_p^{-1} \in C^\infty$$

and $f \circ \psi_p = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2$

Theorem.

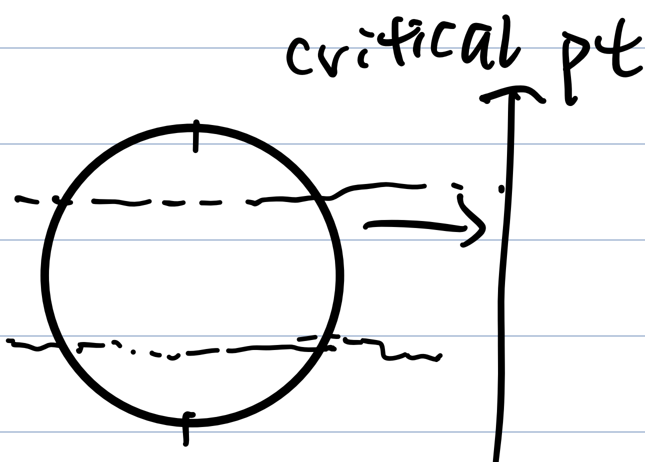
f is a Morse function.

$\{a<f<b\}$ has no critical pt.

Then $\forall\ a<c<d<b$

$\{f \geq c\}$ is homeomorphic to

$\{f \geq d\}$



Cor. critical pt is discrete.

In particular, X is cmpt

$\Downarrow$

critical pts are finite

$\square$

Thm. $\forall$ differential manifold X $\exists$

$\exists$ Morse function f.

Classification. of dim 1 manifold

X. using Morse theory

Find a Morse function f on X

$P_1, \cdots, P_n$ are critical pts.

$\{f(P_1) \leq \cdots \leq f(P_n)\}$

$$\shortparallel$$

$\{c_1 < \cdots < c_m\}$ induction by m.

$X_c = \{f \geq c\} \subseteq X$

Near c_m

(1) $f = c_m + x^2$

(2) $f = c_m - x^2$

Theorem. any compact 1-dimensional Top/Diff manifold must homeo/diffeomorphic to the disjoint union of S^1, $[0,1]$ (finite union)

Dim $=2$:

M.

$\Rightarrow \partial M$ is closed 1-manifold

$\partial M = S^1 \sqcup \cdots \sqcup S^1$

Now we add D^2 to S^1

Consider

$$M \cup_{f_1} D^2 \cup_{f_2} D^2 \cdots\cdots \cup_{f_m} D^2$$

Where $f_i : \begin{array}{c} S^1 \\ \cap \\ D^2 \end{array} \rightarrow C_i \subseteq M$

is a diffeomorphism.

Easier version:

$M \cup_f D$ is homeomorphic

to $M \cup_g D$

pf:

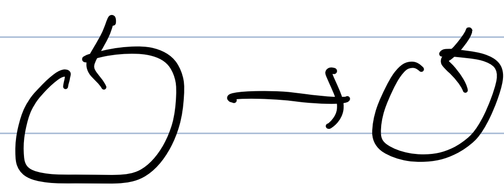
$$F: M \cup_f D \to M \cup_g D$$

$$F(x) = x \quad , \forall x \in M$$

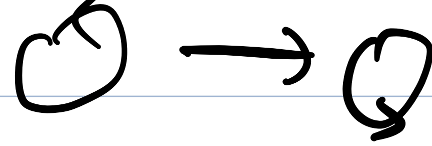
$$F(r,\theta) = (r, g^{-1}(r(\theta)))$$

$$(r,\theta) \in D$$

Lemma. If $f: S^1 \to S^1 \subseteq \mathbb{R}^2$



is a diffeomorphism, $\exists$



$F: S^1 \times [0,1] \to S^1$, s.t.

$F(x,y,0) = f(x,y)$

$F(x,y,1) = (x,y)$ or $(x,-y)$

and $F_t : S^1 \to S^1$

$(x,y) \mapsto F(x,y,t)$

is a diffeomorphism.

pf: $f: S^1 \to S^1$

$[0,1] \xrightarrow{e^{2\pi x i}} S^1 \xrightarrow{f} S^1$

can be lift to $[0,1] \xrightarrow{F} [0,1]$

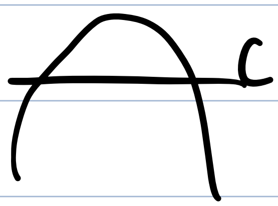
Claim: F must be monotone

If $x < y < z$

$F(x) < F(y)$

$F(y) > F(z)$

$\Rightarrow \exists c$, $F^{-1}(c)$



contains two element

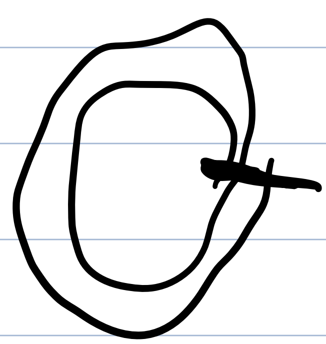
Which is contradict to that f is a diffeomorphism

$\Rightarrow F$ is either $\nearrow$ or $\downarrow$

Moreover

$$F(1) = F(0) \pm 1$$

Otherwise, $|F(1) - F(0)| = m > 1$



X.

If $F(0) = c$

$F(1) = c+1$

We can define

$\overline{F}(x,t) = tx + (1-t) F(x)$

$\Rightarrow \overline{F}(1,t) - \overline{F}(0,t) = 1$

$\Rightarrow \overline{F}$ induce a map

from $S^1 \to S^1$ for each t.

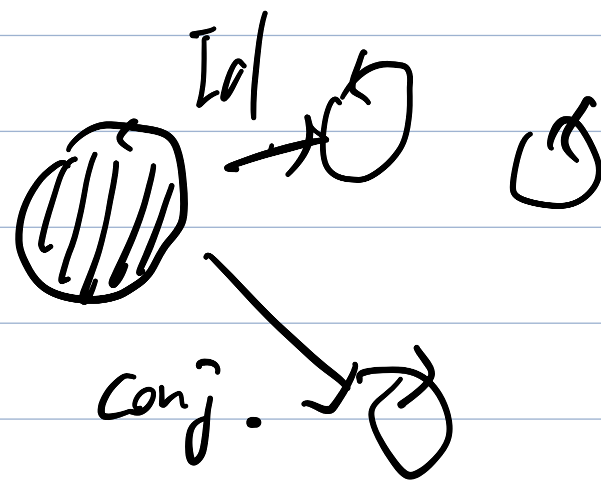
$\overline{F}(x,t)$ is onto by intermidiate value thm.

$F(x,t)$ is onto by it is monotone

$\Rightarrow M \cup_f D$ is diff to

$M \cup_{Id} D$ or

$M \cup_{conj} D$



$Id(x,y) = (x,y)$

$conj(x,y) = (x,-y)$

$$\begin{array}{ccc} M \cup_f D^2 & x \in M & (x,y) \in D^2 \\ \downarrow & \downarrow & \downarrow \\ M \cup_g D^2 & x \in M & (x,-y) \in D^2_- \end{array}$$

extension conjugate map to D^2.

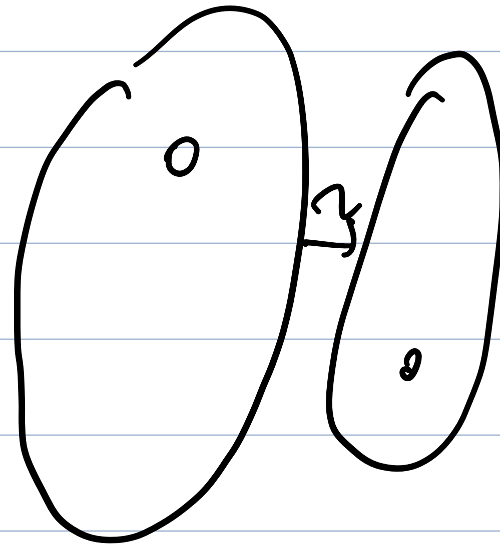
Next question.

Start with a closed manifold

M, u_1, u_2 are coordinate chart, i.e.
$u_1 \cap u_2 \neq \emptyset$.

$\exists f_i : U_i \to V_i \subseteq \mathbb{R}^2$ homeomorphisms,

$f_i \circ f_j^{-1} \in C^\infty$



$D_1 \subseteq V_1 \qquad D_2 \subseteq V_2$

we want to show

$$M - D_1 \xrightarrow[\text{diffe}]{\sim} M - D_2$$

Exercise.

$\{x^2+y^2<10\} \setminus \{x^2+y^2<1\}$

$\Big|$ homeomorphism

$$\{x^2+y^2<10\} \setminus \{x^2+\frac{y^2}{2}<1\}$$

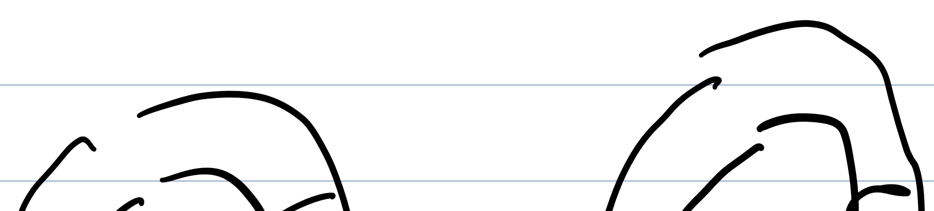
Definition. (Connected sum)

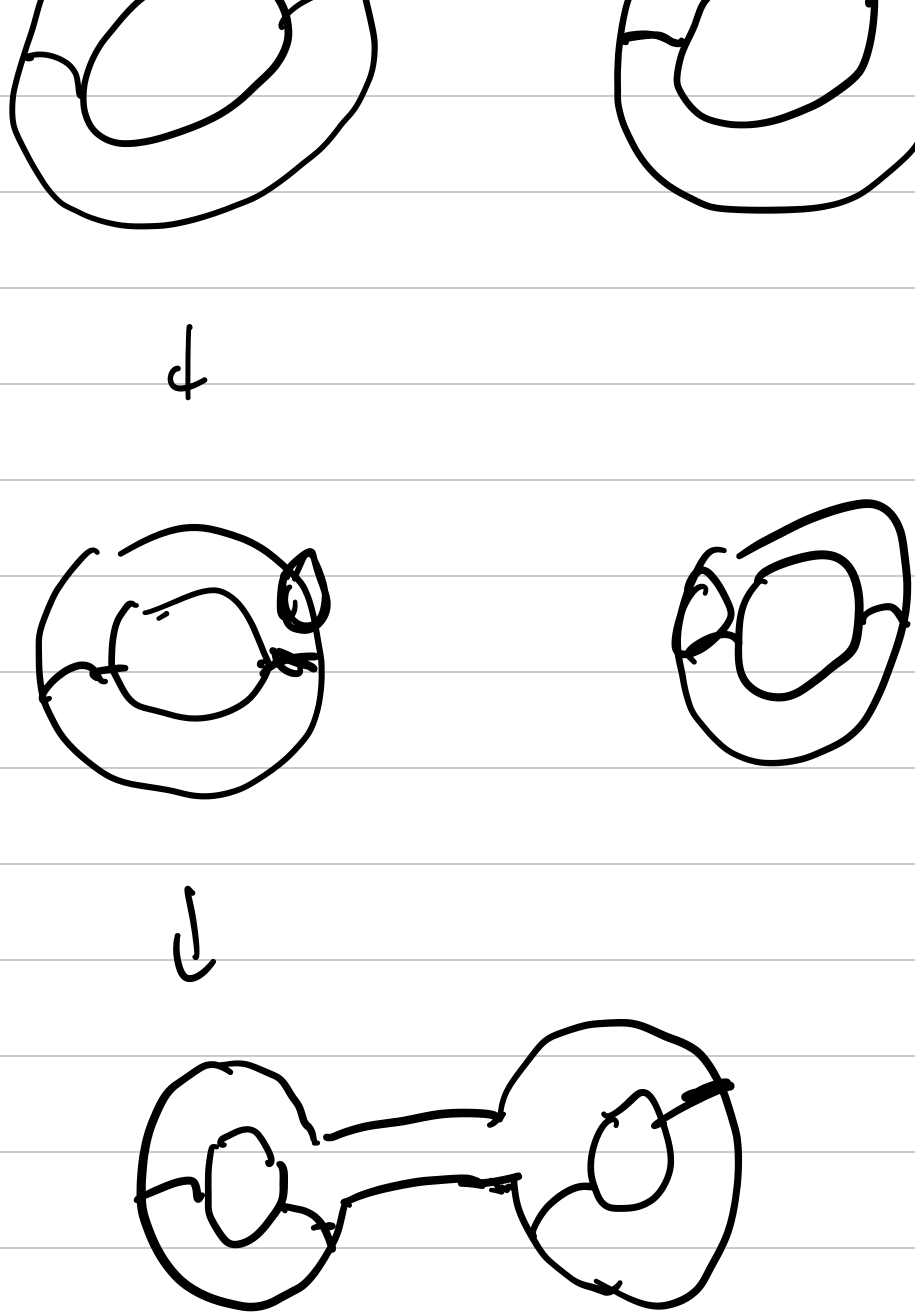
If M_1, M_2 are closed manifold

The connected sum of M_1, M_2

is defined to by

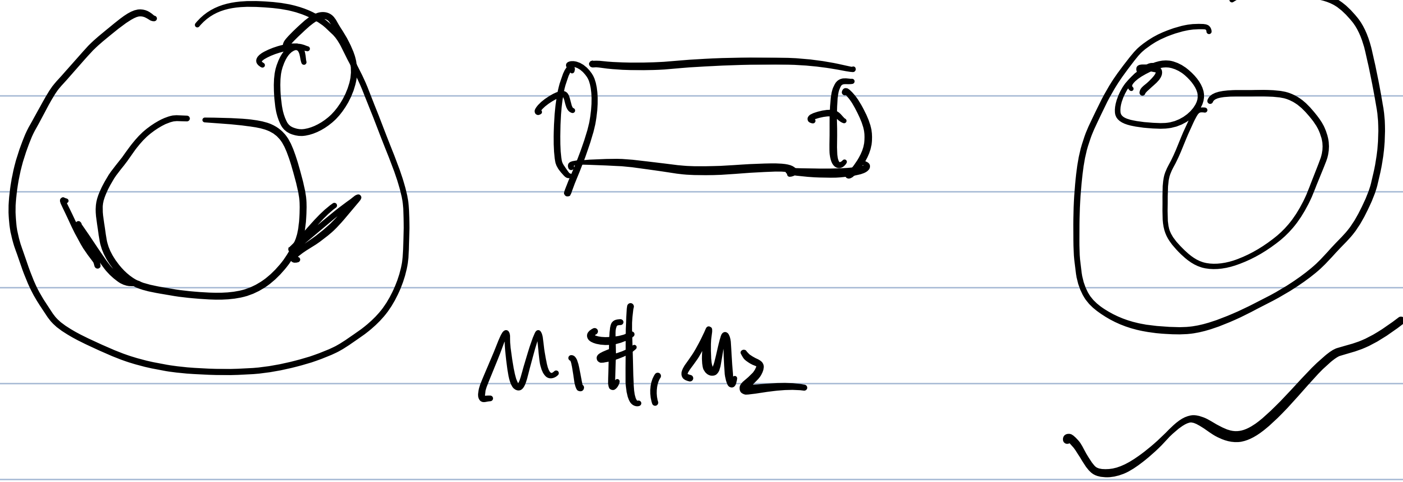
$$(M_1 \setminus D) \cup (S^1 \times [0,1]) \cup (M_2 \setminus D)$$





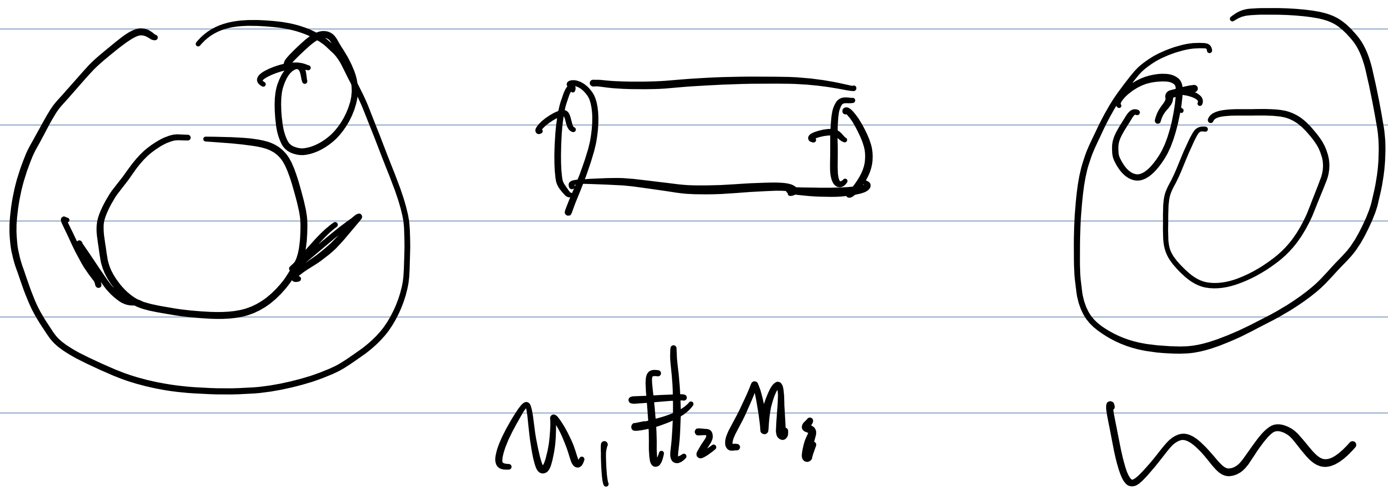
There are two ways of connected sums

M_2



$M_1 \#_1 M_2$

and



$M_1 \#_2 M_2$

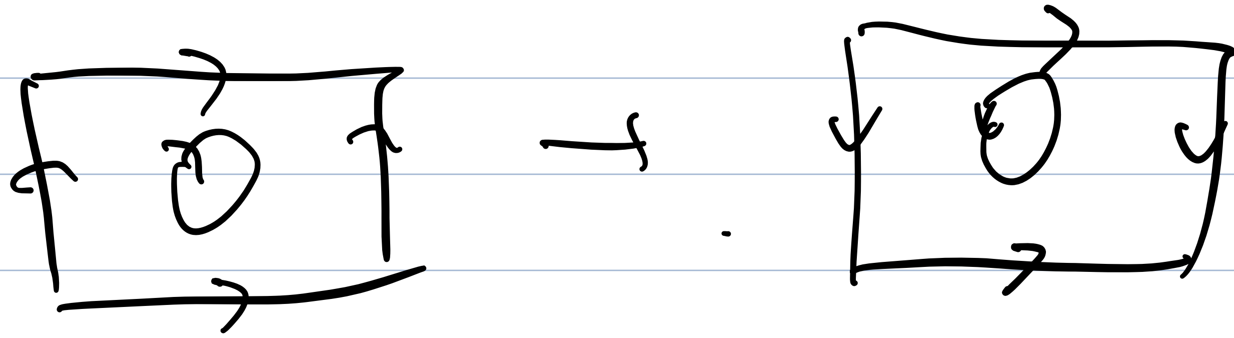
翻转 M_2!

However, if $\exists$ diffe $f: M_1 \to M_1$, s.t.

$f|_{S^1}$ = conj

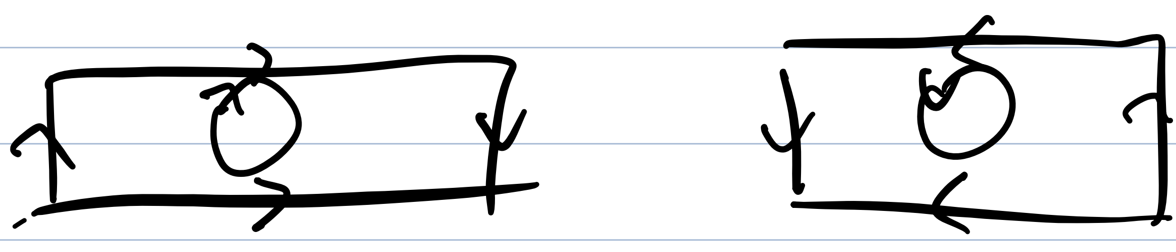
$$\Rightarrow \quad M_1 \#_1 M_2 \xrightarrow{\text{diffe}} M_1 \#_2 M_2$$

Example 1: Torus



Example 2:

Mobius band = $\mathbb{RP}^2 \setminus D$



$(x, y) \longrightarrow (1-x, y)$

For $\mathbb{RP}^2$

$$M \#_1 \mathbb{RP}^2 = M \setminus D \cup_{Id} \text{Mobius}$$

$$= M \setminus D \cup_{conj} \text{Mobius}$$

$$= M \#_2 \mathbb{RP}^2$$

Theorem. Any surface with boundary is homeomorphic to the disjoint union of

m

$$S^2 \# \overbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}$$

$$\# \underbrace{T^2 \# \cdots \# T^2}_{n}$$

removed some disc

Pf: we need to pf this for closed surfaces

If X is closed, f is a morse function.

Critical pt: $P_1 \sim P_m$

$$\{c_1 < \cdots < c_l\} = \{f(p_i)\}$$

(1) $f = c_i + X^2 + Y^2$

(2) $f = c_i + X^2 - Y^2$

(3) $f = c_i - X^2 - Y^2$

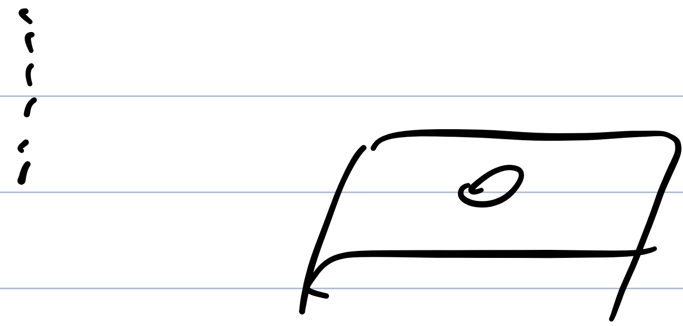
Define $X_c = \{f \geq c\} \subseteq X$

Idea: $X_{c_l + \varepsilon} = \phi$

$X_{c_l - \varepsilon}$

$\vdots$

$$\boxed{X_{c_{l-1} + \varepsilon}}$$



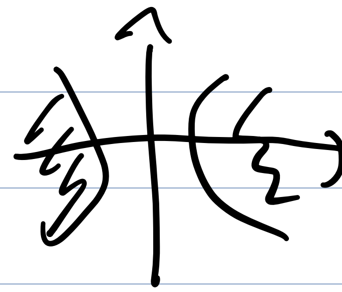
Case 1:

$$\{f \geq c_i + \varepsilon\} = \{x^2+y^2 \geq \varepsilon\}$$
$$= \mathbb{R}^2 \setminus D^2$$

$$\{f \geq c_i - \varepsilon\} = \mathbb{R}^2$$

We get a disc back.

Case 2:

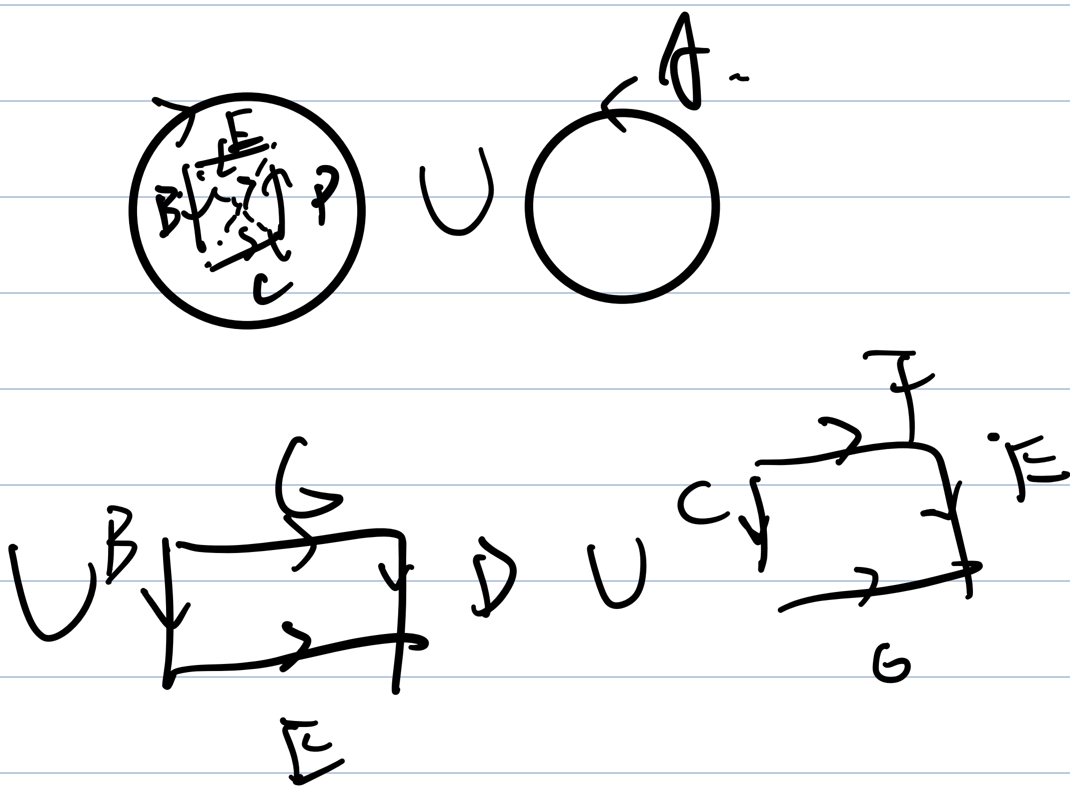


$$\{f \geq c_i + \varepsilon\} = \{\underline{x^2 - y^2 \geq \varepsilon}\}$$

$$\{f \geq c_i - \varepsilon\} = \{x^2 - y^2 \geq -\varepsilon\}$$

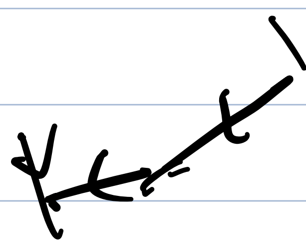
We get a strip.

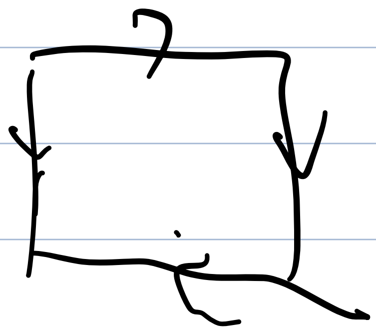
Question. What is



$\|$

RP^2.





Hw: What are they:

(1) $XYXY$

(2) $XYXY^{-1}$

(3) $XYX^{-1}Y$

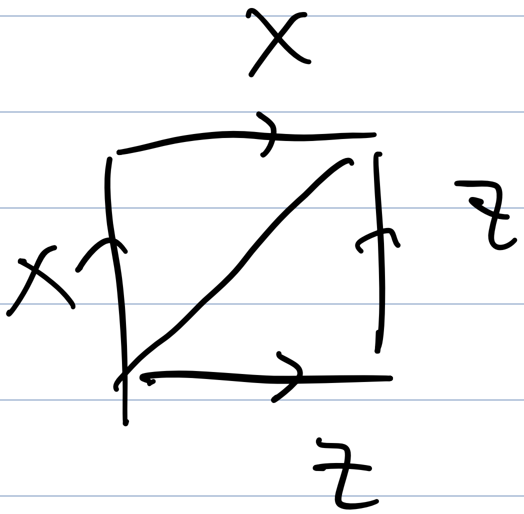
(4) $XYX^{-1}Y^{-1}$

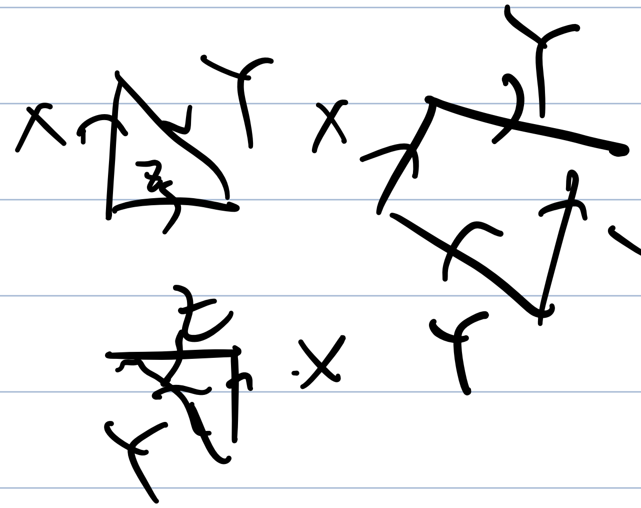
Klein bottle

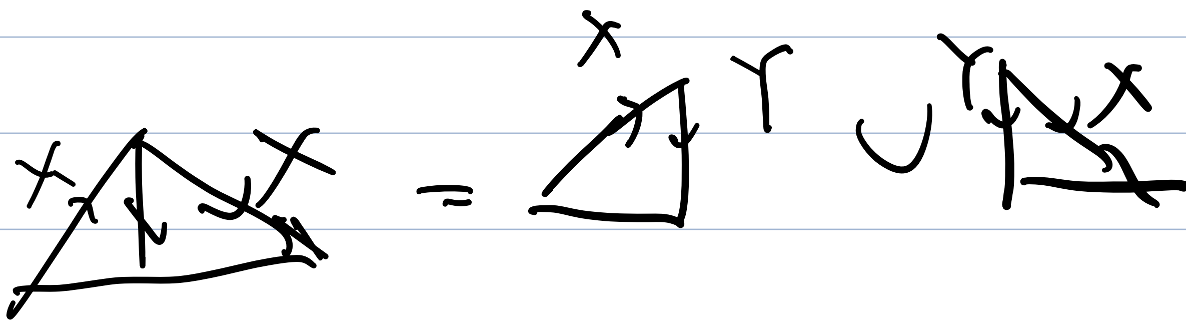
$$= Z^{-1} Y X \qquad ZY = X$$

$$Y = Z^{-1} X$$

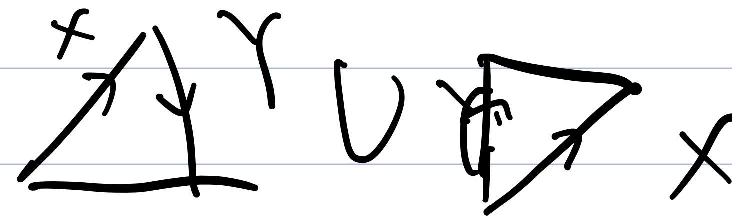
$$= Z^{-1} Z^{-1} X X$$







$=$



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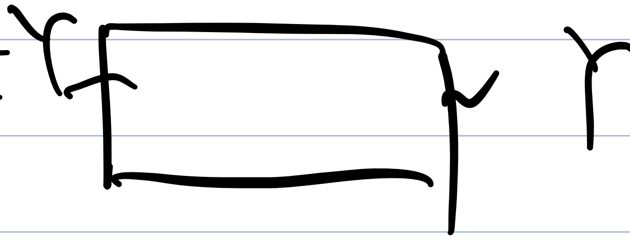
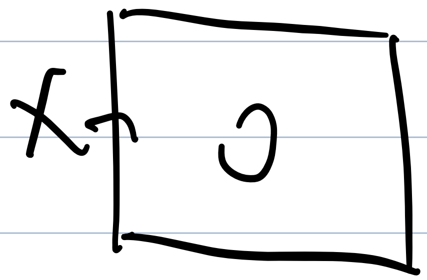


= Mobius

$\Rightarrow$ Klein bottle = 2 Mobius

Mobius = $\mathbb{RP}^2 \setminus$ disc



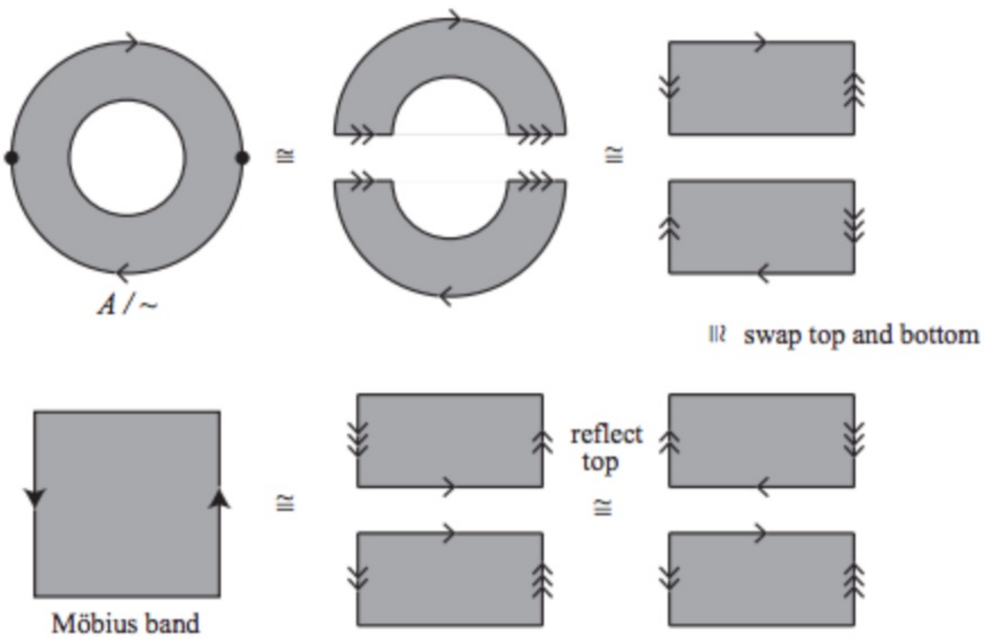
Answers Sorted by: Highest score (default)

4

Let D be the closed unit disk in $\mathbb{R}^2$, and $D/\sim$ the disk with antipodal points on the boundary identified, which is homeomorphic to $\mathbb{RP}^2$.

Now decompose D into an annulus A and a smaller disk, so that attaching a disk to A along the inner circle gives you D.

So, attaching a disk to $A/\sim$ along the inner circle will give you $(D/\sim) \cong \mathbb{RP}^2$. If we can show that $A/\sim$ is homeomorphic to a Möbius band, we're done.



Here's how we do that.

(The image is from the Oxford Part A Topology lecture notes)

$\mathbb{RP}^2$ = disc $\cup$ Mobius band

Klein bottle = 2 Mobius

$S^2 - 2\,\text{disc} \cup 2$ Mobius.

Klein bottle $= \underbrace{S^2}_{X} \# \mathbb{RP}^2 \# \mathbb{RP}^2$

$\mathbb{RP}^2$, Torus, S^2.

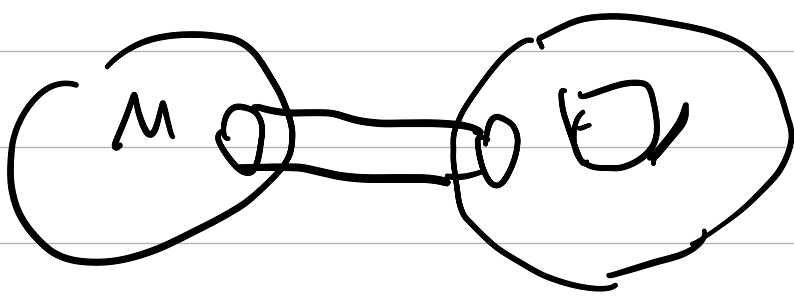
All cmpt surface with boundary

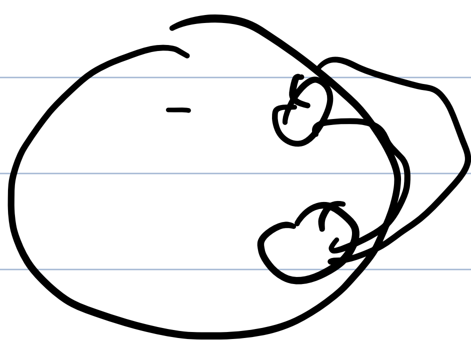
must be a disjoint union of

$S^2 \# \cdots \# \mathbb{RP}^2 \cdots \# \text{Torus} \setminus \text{discs}$

Remark.

$M \# T^2$

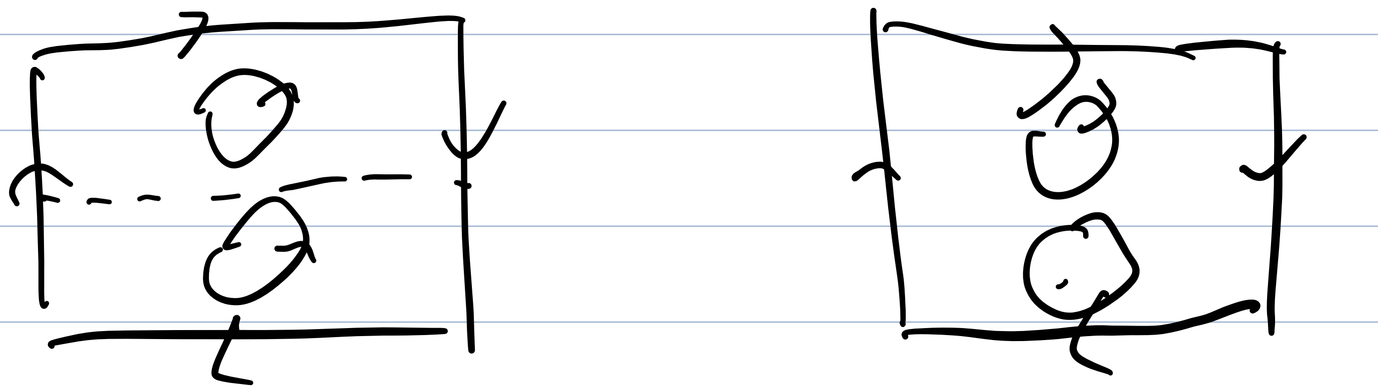
$=$ 

$=$ 

加装正环柄

Klein bottle：反环柄

$\mathbb{RP}^2 \# \text{Torus} = \mathbb{RP}^2 \# \text{Klein bottle}$



$\Rightarrow \mathbb{RP}^2 \# T^2 = \mathbb{RP}^2 \# \text{Klein bottle}$

$= \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$

$\Rightarrow S^2 \# m\mathbb{RP}^2 \# l T^2$

$= S^2 \# (m+2l)\mathbb{RP}^2$ (if $m \neq 0$).

Thm. Any compact surface with boundary must be the disjoint union of

$S^2 \# (m\mathbb{RP}^2) \setminus n$ disc

$S^2 \# (l T^2) \setminus n$ disc

Cor. Any closed connected manifold is

$S^2 \# (m\,\mathbb{RP}^2)$

$S^2 \# (n\,T^2)$

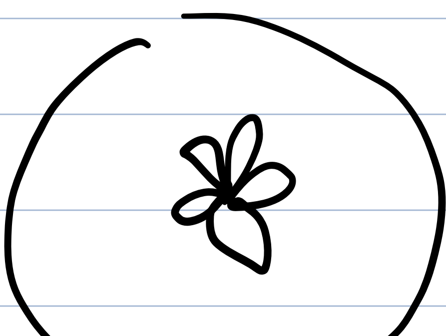
Fundamental group of surface:

$m\,\mathbb{RP}^2$

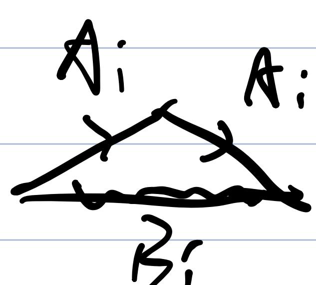
$\mathbb{RP}^2 =$ Mobius $\cup$ disc

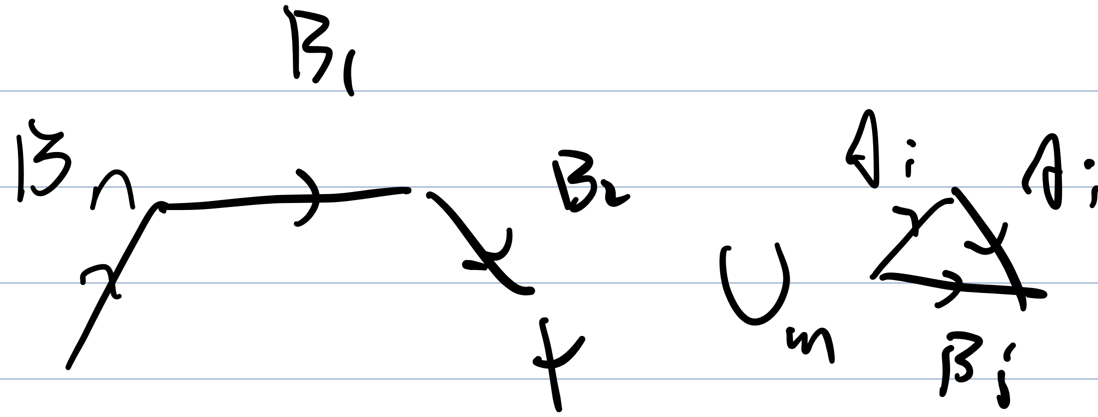
$m\,\mathbb{RP}^2 = S^2 \setminus m$ disc $\cup$ m Mobius

$=$



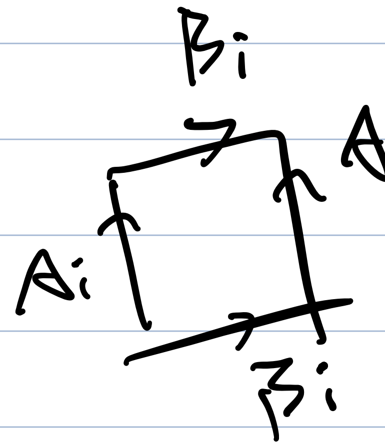
$\bigcup_m$

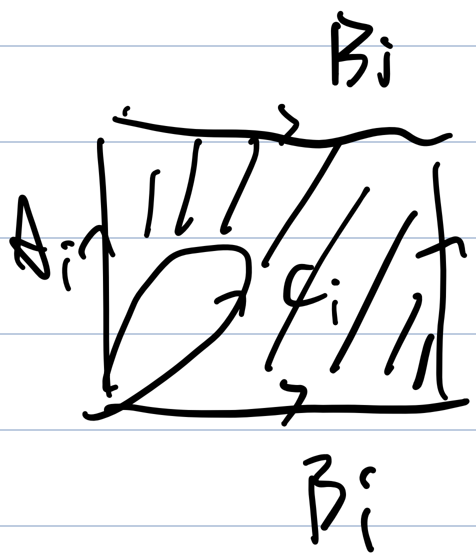


$=$ 

$\cdots$

$= A_1 A_1 A_2 A_2 \cdots A_m A_m$

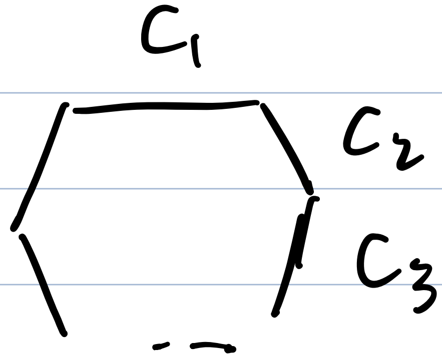




$= T^2\text{-disc}$

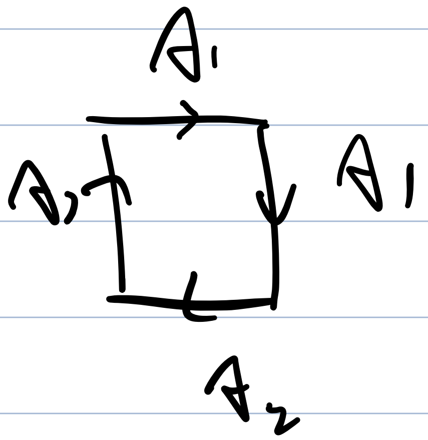
$\|$

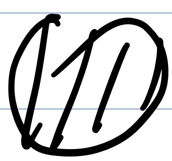
$A_i B_i A_i^{-1} B_i^{-1} C_i$

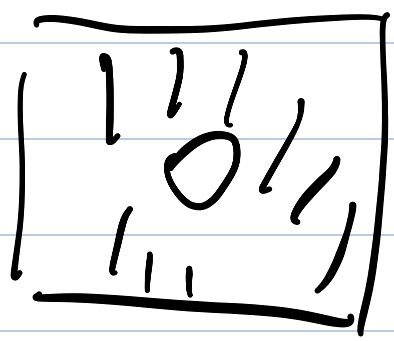
$CT^2 =$ 

$\cup A_i B_i A_i B_i^{-1} C_i$

$= A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \cdots$

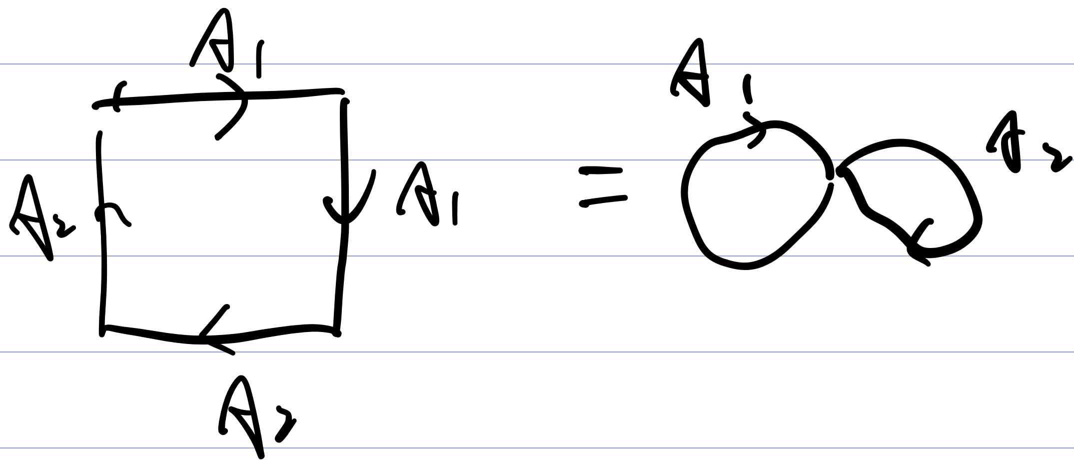


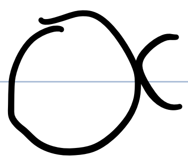
$u =$ 

$V =$ 

$\pi_1(U \cap V) = \mathbb{Z}$

$\pi_1(U) = \{e\}$



$V_1 =$  $\quad V_2 =$ 

$$\pi_1(V) = \pi_1(V_1) * \pi_1(V_2) \Big/ \pi_1(V_1 \cap V_2)$$

$$\simeq \mathbb{Z} * \mathbb{Z}$$

$\Rightarrow \pi_1(U \cup V) = (\mathbb{Z} * \mathbb{Z}) * \{e\} / \mathbb{Z}$

$= \langle A_1, A_2 \mid A_1 A_1 A_2 A_2 = e \rangle$

Similarly,

$\pi_1(m\mathbb{RP}^2)$

$= \langle A_1, \cdots, A_m \mid A_1 A_1 A_2 A_2 \cdots A_m A_m = e \rangle$

$\pi_1(lT^2) = \langle A_1, B_1, \cdots, A_l, B_l \mid A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_l B_l A_l^{-1} B_l^{-1} = e \rangle$

Abelianize $\pi_1 \cdot (H_1)$

$H_1(m\mathbb{RP}^2) = \mathbb{Z}\times\cdots\times\mathbb{Z} \Big/ (2A_1+\cdots+2A_m)$

$H_1(lT^2) = \mathbb{Z}^l$

can't be isomorphic!

$\Rightarrow m\mathbb{RP}^2, lT^2$ have instinct

π_1!

In particular, M is a

M homotopic to S^2

$\Rightarrow M$ is iso. to S^2

Higher dimension:

theorem. any closed manifold with boundary must be homotopic to a CW complex.

Def: X is called a CW-complex, if it's obtained inductively by the following way:

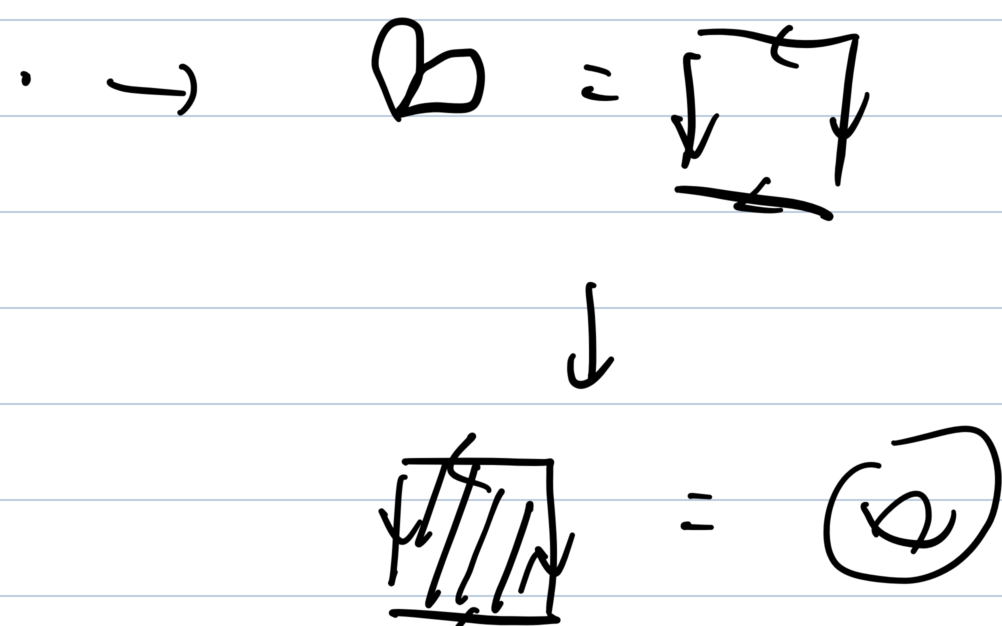
X_0 = several pts.

$$X_i = X_{i-1} \cup_{f_{ij}} B_{i1} \cup \cdots \cup B_{im_i}$$

$$B_{ij} = \{|x|<1\} \subseteq \mathbb{R}^i$$

$$f_{ij} : S_j^{i-1} = \partial B_{ij} \to X_{i-1}$$

Example:



Pf: By Morse theory.

Theorem.

If $f, g : S^{n-1} \to X$

is homotopic to each other

$$\Rightarrow \quad X \cup_f B^n \simeq X \cup_g B^n$$

X